19. SOLUTIONS OF TRIANGLE

1. INTRODUCTION

In any triangle ABC, the side BC, opposite to the angle A is denoted by a; the side CA and AB, opposite to the angles B and C respectively are denoted by b and c respectively. The semi-perimeter of the triangle is denoted by s and its area by Δ or S. In this chapter, we shall discuss various relations between the sides a, b, c and the angles A, B, C of Δ ABC.

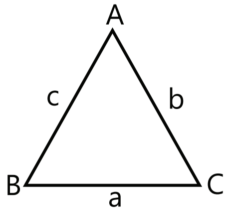

Figure 19.1

2. SINE RULE

The sides of a triangle (any type of triangle) are proportional to the sines of the angle opposite to them in triangle ABC, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$

Note: (i) The above rule can also be written as $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$

(ii) The sine rule is a very useful tool to express the sides of a triangle in terms of sines of the angle and vice-versa in the following manner: $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k(\text{Let});\Rightarrow a=k\sin A,\ b=k\sin B,\ c=k\sin C$

Similarly, $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=\lambda(\text{Let});\ \Rightarrow \sin A=\lambda a,\ \sin B=\lambda b,\ \sin C=\lambda c$

3. COSINE RULE

In any ΔABC, $\cos A=\frac{b^2+c^2-a^2}{2bc}$; $\cos B=\frac{c^2+a^2-b^2}{2ac}$; $\cos C=\frac{a^2+b^2-c^2}{2ab}$

Note: In particular

$\angle A=60^\circ \Rightarrow b^2+c^2-a^2=bc$

$\angle B=60^\circ \Rightarrow c^2+a^2-b^2=ca$

$\angle C=60^\circ \Rightarrow a^2+b^2-c^2=ab$

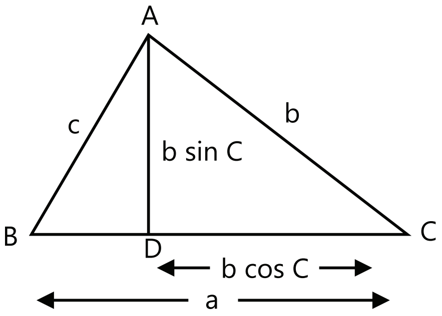

Figure 19.2

4. PROJECTION FORMULAE

If any ΔABC :(i) $a=b\cos C+c\cos B$ (ii) $b=c\cos A+a\cos C$ (iii) $c=a\cos B+b\cos A$

i.e. any side of a triangle is equal to the sum of the projection of the other two sides on it.

Case I: When ΔABC is an acute angled triangle,

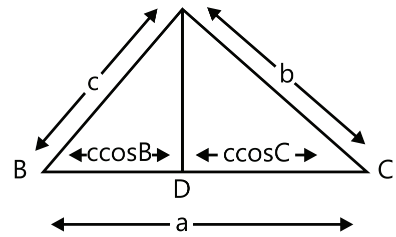

Figure 19.3

$\cos B = \dfrac{BD}{AB} \Rightarrow BD = AB.\cos B \Rightarrow BD = c.\cos B$ and

$\cos C = \dfrac{CD}{AC} \Rightarrow CD = AC.\cos C \Rightarrow CD = b\cos C$

then, BD+DC=BC

$\therefore$ a = c cos B + b cos C

Case II: When ΔABC is an obtuse angled triangle,

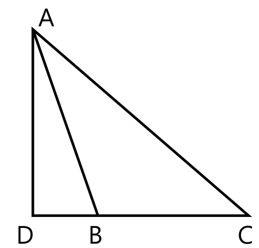

Figure 19.4

$\cos C = \dfrac{CD}{AC} \Rightarrow CD = AC.\cos C$

$CD = b.\cos C$ and $\cos(180 - B) = \dfrac{BD}{AB} \Rightarrow BD = -c.\cos B$ then,

a = BC and CD – BD ⇒ $a = b\cos C + c\cos B$

Illustration 1: If $A = 75^o$, $B = 45^o$, then what is the value of $b + c\sqrt{2}$? **(JEE MAIN)**

Sol: Here, $c = 180^o - 120^o = 60^o$. Therefore by using sine rule, we can solve the above problem.

Use sine rule $\dfrac{a}{\sin 75^o} = \dfrac{b}{\sin 45^o} = \dfrac{c}{\sin 60^o} = K \Rightarrow \begin{matrix} a = k \sin 75^o \\ b = k \sin 45^o \\ c = k \sin 60^o \end{matrix}$

consider, $\left(b + c\sqrt{2}\right) = k\left(\sin 45^o + \sqrt{2}\sin 60^0\right) = k\dfrac{\sqrt{3}+1}{\sqrt{2}} = 2k\dfrac{\sqrt{3}+1}{2\sqrt{2}} = 2k\sin 75^o = 2k\sin A = 2a$

Illustration 2: In a Δ ABC, if $B = 30^o$ and $c = \sqrt{3}b$, then find the value of A. **(JEE MAIN)**

Sol: Here, by using cosine rule $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$ we can easily solve the above problem.

We have $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca} \Rightarrow \dfrac{\sqrt{3}}{2} = \dfrac{3b^2 + a^2 - b^2}{2 \times \sqrt{3}b \times a}$; $\Rightarrow a^2 - 3ab + 2b^2 = 0 \Rightarrow (a-2b)(a-b) = 0$

$\Rightarrow a - b = 0$ OR a – 2b = 0

$\Rightarrow$ Either $a = b \Rightarrow A = 30^o$ or $a = 2b \Rightarrow a^2 \Rightarrow 4b^2 \Rightarrow b^2 + c^2 \Rightarrow A = 90^o$.

Illustration 3: Prove that $a\sin(B - C) + b\sin(C - A) + c\sin(A - B) = 0$ **(JEE MAIN)**

Sol: By sine rule i.e. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$ we can simply prove the above equation.

In equation $a\sin(B - C) + b\sin(C - A) + c\sin(A - B) = 0$, putting a = k sin A, b = k sin B, c = k sin C

= $k\left(\sin A\sin(B - C) + \sin B\sin(C - A) + \sin C\sin(A - B)\right)$ =0 (expanding all terms gets cancelled)

(Using $\sin(\alpha - \beta) = \sin\alpha.\cos\beta - \sin\beta\cos\alpha$)

Illustration 4: Prove that $\sin(B - C) = \dfrac{b^2 - c^2}{a^2}\sin A$ **(JEE MAIN)**

Sol: Given, $\sin(B-C)=\frac{b^2-c^2}{a^2}\sin A \Rightarrow a^2\sin(B-C)=\left(b^2-c^2\right)\sin A$

Takin L.H.S., $a^2\sin(B - C) = a2(\sin B\cos C - \cos B\sin C)$

Now using sine rule, $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=k$ (say) and cosine rule, $\cos B=\frac{a^2+c^2-b^2}{2\,ac}$, and $\cos C=\frac{a^2+b^2-c^2}{2\,ab}$

$$=a^2\left(kb\frac{a^2+b^2-c^2}{2\,ab}-\frac{a^2+c^2-b^2}{2\,ac}\times kc\right)=ka\left(\frac{a^2+b^2-c^2-a^2+c^2-b^2}{2}\right)=\sin A\times\left(b^2-c^2\right)=\text{RHS.}$$

Illustration 5: The angles of a triangle are in 4:1:1 ratio. Find the ratio between its greater side and perimeter? **(JEE ADVANCED)**

Sol: Here, the angles are 120^o, 30^o, 30^o. Therefore, by using sine rule, we will get the required ratio.

Angles are 120^o, 30^o, 30^o.

If the sides opposite to these angles are a, b and c respectively, a will be the greatest side.

Now from sine formula, $\frac{a}{\sin 120^o}=\frac{b}{\sin 30^o}=\frac{c}{\sin 30^o}$; $\Rightarrow \frac{a}{\sqrt{3}/2}=\frac{b}{1/2}=\frac{c}{1/2}$; $\Rightarrow \frac{a}{\sqrt{3}}=\frac{b}{1}=\frac{c}{1}=k$ (say)

then $a=\sqrt{3}k$, perimeter= $\left(2+\sqrt{3}\right)k$; $\therefore$ Required ratio = $\frac{\sqrt{3}k}{\left(2+\sqrt{3}\right)k}=\frac{\sqrt{3}}{2+\sqrt{3}}$.

Illustration 6: Solve $b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}$ in term of k where k is perimeter of the ΔABC. **(JEE ADVANCED)**

Sol: We can solve the given problem simply by $\cos^2\theta=\frac{1+\cos 2\theta}{2}$

Here, $b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}=\frac{b}{2}\left(1+\cos C\right)+\frac{c}{2}(1+\cos B)$ [using projection formula]

$=\frac{b+c}{2}+\frac{1}{2}a=\frac{a+b+c}{2}$; $\therefore b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}=\frac{k}{2}$ [where k=a+b+c, given]

Illustration 7: In any triangle ABC, show that $\frac{a-b}{a+b}=\frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}$ **(JEE ADVANCED)**

Sol: We can derive the values of a, b and c using sine rule and putting it to L.H.S. we can prove the above problem.

We know that, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$

$\Rightarrow a=k\sin A$, $b=k\sin B$, $c=k\sin C$ ...(i)

On putting the values of a and b from (1) on L.H.S., we get

$$\text{L.H.S.}=\frac{a-b}{a+b}=\frac{k\sin A-k\sin B}{k\sin A+k\sin B}=\frac{\sin A-\sin B}{\sin A+\sin B}=\frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}$$

$$=\cot\left(\frac{A+B}{2}\right)\tan\left(\frac{A-B}{2}\right)=\frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}=\text{ R.H.S.}$$

5. NAPIER'S ANALOGY (LAW OF TANGENTS)

In any ΔABC, (i) $\tan\left(\frac{B-C}{2}\right)=\left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$ (ii) $\tan\left(\frac{A-B}{2}\right)=\left(\frac{a-b}{a+b}\right)\cot\left(\frac{C}{2}\right)$ (iii) $\tan\left(\frac{C-A}{2}\right)=\left(\frac{c-a}{c+a}\right)\cot\left(\frac{B}{2}\right)$

Proof:(i) R.H.S. $=\frac{b-c}{b+c}\cot\left(\frac{A}{2}\right)=\frac{k\sin B-k\sin c}{k\sin B+k\sin c}\cot\left(\frac{A}{2}\right)$ [By the sine rule] $=\left(\frac{\sin B-\sin c}{\sin B+\sin c}\right)\cot\left(\frac{A}{2}\right)$

$$=\left(\frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}\right).\cot\left(\frac{A}{2}\right) \text{ [By C \& D formulae]}$$

$$=\tan\left(\frac{B-C}{2}\right).\cot\left(\frac{B+C}{2}\right)\Rightarrow\cot\left(\frac{A}{2}\right)=\tan\left(\frac{B-C}{2}\right).\tan\left(\frac{A}{2}\right).\cot\left(\frac{A}{2}\right) \text{ [By condition identities]}$$

$$=\tan\left(\frac{B-C}{2}\right)=LHS$$

Similarly, (ii) and (iii) can be proved.

Illustration 8: In any triangle ABC, if $A=30^o$, b=3 and $c=3\sqrt{3}$, then find $\angle B$ and $\angle C$. **(JEE MAIN)**

Sol: By using formula, $\tan\frac{C-B}{2}=\frac{c-b}{c+b}\cot\frac{A}{2}$, we can easily obtain the values of $\angle B$ and $\angle C$.

Here $\angle A=30^o$ $\therefore$ $\frac{B+C}{2}=90^o-\frac{A}{2}=90-15^o=75^o$ ... (i)

Since $c>b\Rightarrow\angle C>\angle B$ and $B+C=150^o$ ... (ii)

$$\tan\frac{C-B}{2}=\frac{c-b}{c+b}\cot\frac{A}{2}=\frac{c-b}{c+b}\tan\left(\frac{B+C}{2}\right);\Rightarrow\tan\frac{C-B}{2}=\frac{3\sqrt{3}-3}{3\left(\sqrt{3}+1\right)}\tan 75^o$$

$$\Rightarrow\tan\left(\frac{C-B}{2}\right)=\frac{c-b}{c+b}\cot\left(\frac{\pi}{2}-A\right)=\frac{c-b}{c+b}\tan\left(\frac{A}{2}\right)$$

[Using (1)] $\Rightarrow\tan\frac{C-B}{2}=\frac{3\left(\sqrt{3}-1\right)}{3\left(\sqrt{3}+1\right)}\tan\left(45^o+30^o\right)=\frac{\left(\sqrt{3}-1\right)}{\left(\sqrt{3}+1\right)}\left(\frac{\tan 45^o+\tan 30^o}{1-\tan 45^o\tan 30^o}\right)$

$$=\frac{\left(\sqrt{3}-1\right)}{\left(\sqrt{3}+1\right)}\left(\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}\right)=\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)=1\ ;\Rightarrow\frac{C-B}{2}=45^o\ ;\left[\because\tan 45^o=1\right]$$

$\Rightarrow C-B=90^o$ ... (iii)

Solving (ii) and (iii), we get $\angle B=30^o$ and $\angle C=120^o$

6. TRIGONOMETRIC RATIOS OF HALF ANGLES

Sine, cosine and tangent of half the angles of any triangle are related to their sides as given below. Note that the perimeter of ΔABC will be denoted by 2s i.e. 2s=a+b+c and the area denoted by Δ.

Formulae for $\sin\left(\frac{A}{2}\right), \sin\left(\frac{B}{2}\right), \sin\left(\frac{C}{2}\right)$ for any ΔABC

(i) $\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\left(\frac{B}{2}\right) = \sqrt{\frac{(s-c)(s-a)}{ac}}$ (iii) $\sin\left(\frac{C}{2}\right) = \sqrt{\frac{(s-a)(s-b)}{ab}}$

Formulae for $\cos\left(\frac{A}{2}\right), \cos\left(\frac{B}{2}\right), \cos\left(\frac{C}{2}\right)$ for any ΔABC

(i) $\cos\left(\frac{A}{2}\right) = \sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\left(\frac{B}{2}\right) = \sqrt{\frac{s(s-b)}{ac}}$ (iii) $\cos\left(\frac{C}{2}\right) = \sqrt{\frac{s(s-c)}{ab}}$

Formulae for $\tan\left(\frac{A}{2}\right), \tan\left(\frac{B}{2}\right), \tan\left(\frac{C}{2}\right)$ for any ΔABC

(i) $\tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ (ii) $\tan\left(\frac{B}{2}\right) = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$ (iii) $\tan\left(\frac{C}{2}\right) = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Illustration 9: In a triangle ABC, if $\frac{s-a}{11} = \frac{s-b}{12} = \frac{s-c}{13}$, then find the value of $\tan^2(A/2)$. **(JEE MAIN)**

Sol: As we know, $\tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$. Therefore, by using this formula, we can solve the above problem.

$\frac{s-a}{11} = \frac{s-b}{12} = \frac{s-c}{13} = \frac{3s-(a+b+c)}{11+12+13} = \frac{s}{36}$; Now $\tan^2\left(\frac{A}{2}\right) = \frac{(s-b)(s-c)}{s(s-a)} = \frac{12\times 13}{36\times 11} = \frac{13}{33}$

Illustration 10: In a triangle ABC, prove that $(a+b+c)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right) = 2c\cot\frac{C}{2}$. **(JEE MAIN)**

Sol: Here by using $\tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ and $\tan\left(\frac{B}{2}\right) = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$ we can prove the above problem.

$$\text{L.H.S.} = (a+b+c)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right) = 2s\left[\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}\right] = 2s\sqrt{\frac{s-c}{s}}\left[\sqrt{\frac{s-b}{s-a}} + \sqrt{\frac{s-a}{s-b}}\right]$$

$$= 2s\sqrt{\frac{(s-c)}{s}}\left[\frac{s-b+s-a}{\sqrt{s-a}\sqrt{s-b}}\right] = \frac{2\sqrt{s}\sqrt{s-c}}{\sqrt{s-a}\sqrt{s-b}}(a+b+c-b-a) = 2c\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \frac{2c}{\tan\frac{C}{2}} = 2c\cot\frac{C}{2} = \text{R.H.S.}$$

Alternate: L.H.S. = $2s\left[\frac{\Delta}{s(s-a)} + \frac{\Delta}{s(s-b)}\right]$

$$= 2\Delta\left[\frac{1}{s-a} + \frac{1}{s-b}\right] = \frac{2\Delta(2s-a-b)}{(s-a)(s-b)} = \frac{2c\Delta^2}{(s-a)(s-b)\Delta} = \frac{2cs(s-c)}{\Delta} = 2c.\cot\frac{c}{2} = \text{R.H.S.}$$

Illustration 11: In a ΔABC, if $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ are in AP, then prove that the sides of ΔABC are in A.P. **(JEE MAIN)**

Sol: Here by using trigonometric ratios of half angles formula, we can prove the above illustration.

Given $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ are in A.P.

$\Rightarrow \frac{s(s-a)}{\Delta}, \frac{s(s-b)}{\Delta}, \frac{s(s-c)}{\Delta}$ are in A.P.

$\Rightarrow (s-a), (s-b), (s-c)$ are in A.P.

$\Rightarrow$ a, b, c are in A.P. **Proved**

Illustration 12: In a ΔABC, the sides a, b and c are in A.P. Then what is the value of $\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right):\cot\frac{B}{2}$? **(JEE ADVANCED)**

Sol: Simply by using formula of $\tan\frac{A}{2}$, $\tan\frac{C}{2}$ and $\cot\frac{B}{2}$ we can easily get the required result.

$$\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right):\cot\frac{B}{2} \Rightarrow \left[\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}+\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}\right]:\sqrt{\frac{s(s-b)}{(s-c)(s-a)}} \Rightarrow \frac{(s-c)+(s-a)}{\sqrt{s}}:\sqrt{s}$$

$= 2s-(a+c):s;\ \Rightarrow b:\frac{a+b+c}{2};\ \Rightarrow 2b:a+b+c \Rightarrow$ a.b.c are in A.P.

$\therefore 2b{:}a+b+c = 2:3$

Illustration 13: In any triangle ABC, show that $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{a+b+c}{a+b-c}\cot\frac{C}{2}$. **(JEE ADVANCED)**

Sol: Similar to the above problem, by putting the values of $\cot\frac{A}{2}$, $\cot\frac{B}{2}$ and $\cot\frac{C}{2}$ we can prove the above problem.

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}+\sqrt{\frac{s(s-b)}{(s-c)(s-a)}}+\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$\text{L.H.S.}\frac{1}{\sqrt{s(s-a)(s-b)(s-c)}}\left(\sqrt{s^2(s-a)^2}+\sqrt{s^2(s-b)^2}+\sqrt{s^2(s-c)^2}\right)=\frac{1}{\sqrt{s(s-a)(s-b)(s-c)}}\left[s(s-a+s-b+s-c)\right]$$

$$=\frac{1}{\sqrt{s(s-a)(s-b)(s-c)}}\left[s\{3s-(a+b+c)\}\right]=\frac{1}{\sqrt{s(s-a)(s-b)(s-c)}}\left[s(3s-2s)\right]$$

$$\Rightarrow \cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s^2}{\sqrt{s(s-a)(s-b)(s-c)}} \quad \text{... (i)}$$

Now, R.H.S. $=\frac{a+b+c}{a+b-c}\cot\frac{C}{2}=\frac{2s}{a+b+c-2c}\cot\frac{C}{2}=\frac{2s}{2s-2c}\cot\frac{C}{2}=\frac{s}{s-c}\cot\frac{C}{2}$

L.H.S = R.H.S

$$=\frac{s}{s-c}\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}=\frac{s^2}{\sqrt{s(s-a)(s-b)(s-c)}};\Rightarrow \frac{a+b+c}{a+b-c}\cot\frac{C}{2}=\frac{s^2}{\sqrt{s(s-a)(s-b)(s-c)}} \quad \text{... (ii)}$$

From (i) and (ii), we have $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{a+b+c}{a+b-c}\cot\frac{C}{2}$. **Proved**

7. AREA OF TRIANGLE

If Δ be the area of a triangle ABC, then $\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$

Proof: Let ABC be a triangle. Then the following cases arise.

Case I: When ΔABC is an acute angled triangle, $\sin B = \frac{AD}{AB}$

AD=AB sin B ; AD=c sin B; $\Delta = \text{Area of } \Delta ABC$; $\Delta = \frac{1}{2}(BC)(AD)$; $\Delta = \frac{1}{2}ac\sin B$

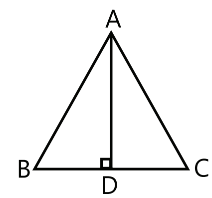

Figure 19.5

Case-II: When Δ ABC is an obtuse angled triangle, $\sin(180 - B) = \frac{AD}{AB}$;

AD=AB sin B $\Rightarrow$ AD=c sin B

Δ =Area of ΔABC ; $\Delta = \frac{1}{2}(BC)(AD)$; $\Delta = \frac{1}{2}ac\sin B$; So in each case, $\Delta = \frac{1}{2}ac\sin B$

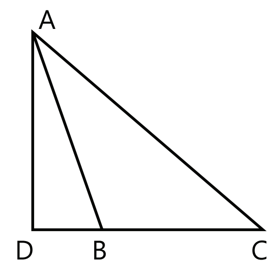

Figure 19.6

(ii) Heron's formula $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

Proof: $\Delta = \frac{1}{2}bc\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right) = bc\sin\left(\frac{A}{2}\right).\cos\left(\frac{A}{2}\right)$

$= bc\sqrt{\frac{(s-b)(s-c)}{bc}} \times \sqrt{\frac{s(s-a)}{bc}}$ [By half angle formula] = $\sqrt{s(s-a)(s-b)(s-c)}$

(iii) $\Delta = \frac{1}{2}\frac{a^2\sin B\sin C}{\sin(B+C)} = \frac{1}{2}\frac{b^2\sin C\sin A}{\sin(C+A)} = \frac{1}{2}\frac{c^2\sin A\sin B}{\sin(A+B)}$

From the above results, we obtain the following values of sinA, sinB and sinC

(iv) $\sin A = \frac{2\Delta}{bc} = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$ (v) $\sin B = \frac{2\Delta}{ca} = \frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)}$

(vi) $\sin C = \frac{2\Delta}{ab} = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$

Further with the help of (iv), (v), (vi) we obtain $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2\Delta}{abc}$

Illustration 14: In any triangle ABC, prove that $4\Delta\cot A = b^2 + c^2 - a^2$. **(JEE MAIN)**

Sol: We can prove the above problem by using formula of area of triangle i.e. $\Delta = \frac{1}{2}bc\sin A$ and $\cos A = \frac{b^2+c^2-a^2}{2bc}$.

L.H.S.= $4\Delta\cot A = 4.\frac{1}{2}bc\sin A.\frac{\cos A}{\sin A} = 2bc\cos A = 2bc.\frac{b^2+c^2-a^2}{2bc} = b^2+c^2-a^2$ = R.H.S.

Illustration 15: In any triangle ABC, prove that $\frac{a^2-b^2}{2}.\frac{\sin A\sin B}{\sin(A-B)} = \Delta$ **(JEE MAIN)**

Sol: By putting $a = k\sin A$ and $b = k\sin B$ we can prove the above illustration.

L.H.S. $= \frac{a^2-b^2}{2}.\frac{\sin A\sin B}{\sin(A-B)} = \frac{(k^2\sin^2 A - k^2\sin^2 B)\sin A\sin B}{2\sin(A-B)} = \frac{k^2(\sin^2 A - \sin^2 B)\sin A\sin B}{2\sin(A-B)}$

[using sine formula a=k sinA etc.]= $\frac{k^2\sin(A+B)\sin(A-B)\sin A\sin B}{2\sin(A-B)} = \frac{k^2}{2}.\sin(A+B)\sin A\sin B$

$=\frac{1}{2}(k\sin A)(k\sin B)\sin(\pi-c)$; $[\because A+B=\pi-C]$; $\frac{1}{2}ab\sin C=\Delta=$ R.H.S.

Illustration 16: A tree stands vertically on a hill side which make an angle of 15° with the horizontal. From a point on the ground 35m down the hill from the base of the tree, the angle of elevation of the top of the tree is 60°. Find the height of the tree. **(JEE MAIN)**

Sol: We can simply obtain the height of the tree from the given figure.

In ΔAQR, $QR=AQ\sin 15^o=35\sin 15^o$; $AR=AQ\cos 15^o=35\cos 15^o$

In ΔAPR, $\tan 60^o=\frac{PR}{AR}$; $\Rightarrow PR=AR.\sqrt{3}$; $\Rightarrow PQ+QR=\sqrt{3}AR$

$\Rightarrow h+35\sin 15^o=\sqrt{3}.35\cos 15^o$

$\Rightarrow h=35\left(\sqrt{3}\cos 15^o-\sin 15^o\right)=35\left(\sqrt{3}.\frac{\sqrt{3}+1}{2\sqrt{2}}-\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$

$=35\left(\frac{3+\sqrt{3}-\sqrt{3}+1}{2\sqrt{2}}\right)=\frac{35}{2\sqrt{2}}(4)=35\sqrt{2}m$

Hence, the height of the tree= $35\sqrt{2}m$

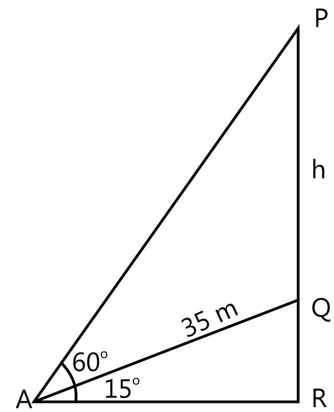

Figure 19.7

Illustration 17: In any triangle ABC, prove that $a\cos A+b\cos B+c\cos C=2a\sin B\sin C=\frac{8\Delta^2}{abc}$. **(JEE ADVANCED)**

Sol: we can solve this illustration by substituting $a=k\sin A$, $b=k\sin B$, and $c=k\sin C$.

As $a=k\sin A$, $b=k\sin B$, $c=k\sin C$ ...(i)

L.H.S.= $a\cos A+b\cos B+c\cos C=k\sin A\cos A+k\sin B\cos B+k\sin C\cos C$ [using sine formula]

$=\frac{k}{2}\left[\{\sin 2A+\sin 2B\}+\sin 2C\right]=\frac{k}{2}\left[2\sin(A+B)\cos(A-B)+2\sin C\cos C\right]$

$=\frac{k}{2}\left[2\sin C\cos(A-B)+2\sin C\cos C\right]=\frac{k}{2}\left[2\sin C\left\{\cos(A-B)+\cos\left(\pi-\overline{A+B}\right)\right\}\right]$

$=k\sin C\left[\cos(A-B)-\cos(A+B)\right]=k\sin C\left[2\sin A\sin B\right]$

$=2(k\sin A).\sin B\sin C=2a\sin B\sin C=2a\left(\frac{2\Delta}{ac}\right)\left(\frac{2\Delta}{ab}\right)$; $\left[\Delta=\frac{1}{2}ac\sin B\Rightarrow\sin B=\frac{2\Delta}{ac}\text{ and }\sin C=\frac{2\Delta}{ab}\right]$

$=\frac{8\Delta^2}{abc}$ =R.H.S.

Illustration 18: The angle of elevation of the top point P of the vertical tower PQ of height h from a point A is 45° and from a point B, the angle of elevation is 60°, where B is a point at a distance d from the point A measured along the line AB which makes an angle 30^o with AQ. Prove that $h=d\left(\sqrt{3}-1\right)$ **(JEE ADVANCED)**

Sol: By using sine rule in ΔABP, we can prove that $h=d\left(\sqrt{3}-1\right)$

In the figure, PQ represents a tower of height h. The angle of elevation of the point P from the point A on the ground is

$\Rightarrow \angle PAQ = 45^o$; $\Rightarrow \angle PAB + \angle BAQ = 45^o$; $\Rightarrow \angle PAB + 30^o = 45^o$

$\angle PAB = 45^o - 30^o$ [Given $\angle BAQ = 30^o$]

$\angle PAB = 15^o$ ... (i)

(Given) $\angle APH = 45^o$; $\Rightarrow \angle APB + \angle BPH = 45^o$ (given) $\Rightarrow \angle APB + 30 = 45^o \Rightarrow \angle APB = 15^o$ ... (ii)

From (i) and (ii), we have $\angle PAB = \angle APB$ So BP=AB=d; $\Rightarrow BP = d$

[Given AB=d]

Again $\angle PAQ = 45^o$, $\angle Q = 90^o \Rightarrow \angle APQ = 45^o$

In ΔAPQ, $\angle PAQ = \angle APQ \Rightarrow AQ = PQ = h$

$AP^2 = PQ^2 + AQ^2 = h^2 + h^2 \Rightarrow AP^2 = 2h^2 \Rightarrow AP = \sqrt{2}h$

Applying sine formula in ΔABP, we get $\frac{AB}{\sin 15^o} = \frac{AP}{\sin 150^o}$

$$\Rightarrow \frac{d}{\sin 15^o} = \frac{\sqrt{2}h}{\sin 150^o} \Rightarrow d = \frac{\sqrt{2}h}{\frac{1}{2}}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) \Rightarrow d = \left(\sqrt{3}-1\right)h$$

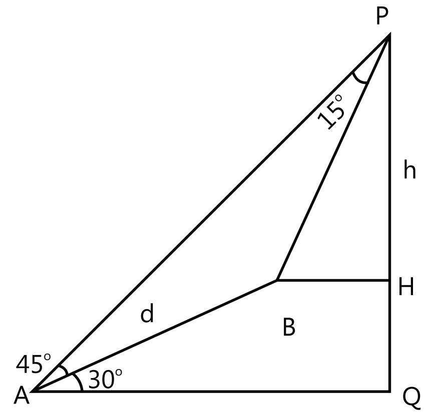

Figure 19.8

Illustration 19: A lamp post is situated at the middle point M of the side AC of a triangular plot ABC with BC=7m, CA=8m and AB=9m. This lamp post subtends an angle $\tan^{-1}(3)$ at the point B. Determine the height of the lamp post. **(JEE ADVANCED)**

Sol: Here in $\Delta BMP \Rightarrow \tan\angle PBM = \frac{PM}{BM}$, therefore by obtaining the value of BM we can find out the height of lamp post.

Here, ABC is a triangular plot. A lamp post PM is situated at the mid-point M of the side AC. Here PM subtends an angle $\tan^{-1}(3)$ at the point B. a=7m, b=8m and c=9m

In ΔABC, $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ or $\cos C = \frac{BC^2 + CA^2 - AB^2}{2BC.CA}$

$$\cos C = \frac{7^2 + 8^2 - 9^2}{2 \times 7 \times 8} = \frac{49 + 64 - 81}{112} = \frac{32}{112} = \frac{2}{7} \qquad \text{... (i)}$$

In ΔBCM, $\cos C = \frac{BC^2 + CM^2 - BM^2}{2BC.CM}$; $\cos C = \frac{7^2 + 4^2 - BM^2}{2 \times 7 \times 4} = \frac{65 - BM^2}{56}$

$\Rightarrow \frac{2}{7} = \frac{65 - BM^2}{56} \Rightarrow 65 - BM^2 = \frac{2}{7} \times 56 = 16$ [Using(i)]

$\Rightarrow BM^2 = 65 - 16 = 49 \Rightarrow BM = 7m$

In $\Delta BMP \Rightarrow \tan\angle PBM = \frac{PM}{BM} \Rightarrow \tan\left(\tan^{-1} 3\right) = \frac{PM}{BM} \Rightarrow 3 = \frac{PM}{7} \Rightarrow PM = 21m$

Hence, the height of the lamp post =21m.

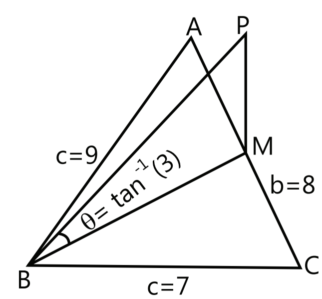

Figure 19.9

8. PROPERTIES OF TRIANGLE

8.1 Circumcircle

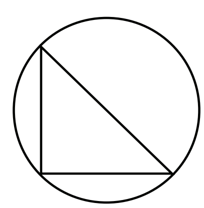

Figure 19.10

A circle passing through the vertices of a triangle is called a circumcircle of the triangle. The centre of the circumcircle is called the circumcentre of the triangle and it is the point of intersection of the perpendicular bisectors of the sides of the triangle. The radius of the circumcircle is called the circumradius of the triangle and is usually denoted by R and is given by the following formulae: $R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=\frac{abc}{4\Delta S}$ Where Δ is area of triangle and $s=\frac{a+b+c}{2}$.

8.2 Incircle

The circle which can be inscribed within the triangle so as to touch all the three sides of the triangle is called the incircle of the triangle. The centre of the incircle is called the incentre of the triangle and it is the point of intersection of the internal bisectors of the angles of the triangle. The radius of the circle is called the inradius of the triangle and is usually denoted by rin-Radius: The radius r of the inscribed circle of a triangle ABC is given by

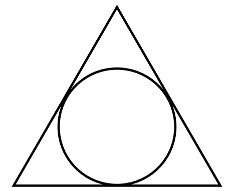

Figure 19.11

(a) $r=\frac{\Delta}{s}$ (ii) $r=(s-a)\tan\left(\frac{A}{2}\right)$, $r=(s-b)\tan\left(\frac{B}{2}\right)$ and $r=(s-c)\tan\left(\frac{C}{2}\right)$

(b) $r=\frac{a\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$, $r=\frac{b\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{B}{2}\right)}$ and $r=\frac{c\sin\left(\frac{B}{2}\right)\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{C}{2}\right)}$

(c) $r=4R\sin\left(\frac{A}{2}\right).\sin\left(\frac{B}{2}\right).\sin\left(\frac{C}{2}\right)$

8.3 Centroid

In ΔABC, the mid-points of the sides BC, CA and AB are D,E and F respectively. The lines AD, BE and CF are called medians of the triangle ABC. The points of concurrency of three medians is called the centroid. Generally it is represented by G.

Also, $AG=\frac{2}{3}AD, BG=\frac{2}{3}BE$ and $CG=\frac{2}{3}CF$.

Length of medians from Figure 9.12

$$\Rightarrow AD^2=b^2+\frac{a^2}{4}-ab\left(\frac{b^2+a^2-c^2}{2ab}\right)$$

$$\Rightarrow AD^2=\frac{2b^2+2c^2-a^2}{4}\Rightarrow AD=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$$

Similarly, $BE=\frac{1}{2}\sqrt{2a^2+2c^2-b^2}$ and $CF=\frac{1}{2}\sqrt{2a^2+2b^2-c^2}$

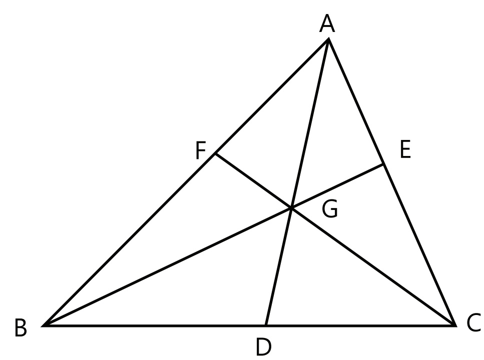

Figure 19.12

8.4 Apollonius Theorem

$$AB^2+AC^2=2\left(AD^2+BD^2\right)$$

Proof: $2\left(AD^2+BD^2\right)=2\left[\frac{1}{2}\left(2b^2+2c^2-a^2\right)+\frac{a^2}{4}\right]=b^2+c^2=AB^2+AC^2$

8.5 Orthocentre

The point of intersection of perpendiculars drawn from the vertices on the opposite sides of a triangle is called its orthocentre.Let the perpendicular AD,BE and CF from the vertices A,B and C on the opposite sides BC,CA and AB of ABC, respectively meet at O.

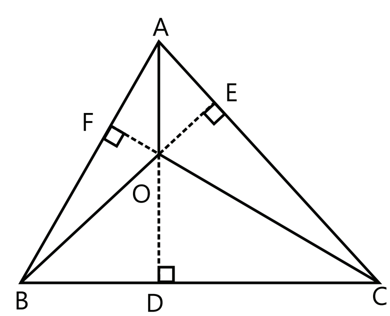

Figure 19.13

Then O is the orthocentre of the ΔABC. The triangle DEF is called the pedal Triangle of the ΔABC.

Centroid (G) of a triangle is situated on the line joining its circumcentre (O) and orthocenter (H) show that the line divides joining its circumcentre (O) and orthocenter (H) in the ratio 1:2.

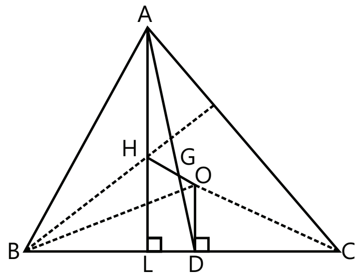

Figure 19.14

Proof: Let AL be a perpendicular from A on BC, then H lies on AL. If OD is perpendicular from O on BC, then D is mid-point of BC.

$\therefore$ AD is a median of ΔABC. Let the line HO meet the median AD at G. Now, we shall prove that G is the centroid of the ΔABC. Obviously, ΔOGD and ΔHGA are similar triangles.

$$\therefore \frac{OG}{HG} = \frac{GD}{GA} = \frac{OD}{HA} = \frac{R\cos A}{2R\cos A} = \frac{1}{2}$$

$\therefore GD = \frac{1}{2}GA \Rightarrow G \Rightarrow$ is centroid of ΔABC and $OG : HG = 1 : 2$

The distances of the orthocenter from the vertices and the sides: If O is the orthocenter and DEF the pedal triangle of the ΔABC, where AD, BE, CF are the perpendiculars drawn from A,B,C on the opposite sides BC,CA,AB respectively, then

(i) OA = 2R cosA, OB = 2R cosB and OC = 2R cosC

(ii) OD = 2R cosBcosC, OE=2R cosCcosA and OF =2R cosAcosB, where R is circumradius.

(iii) The circumradius of the pedal triangle = $\frac{R}{2}$

(iv) The area of pedal triangle $= 2\Delta\cos A\cos B\cos C$.

(v) The sides of the pedal triangle are $a\cos A$, $b\cos B$ and $c\cos C$ and its angles are $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$.

(vi) Circumradii of the triangles OBC, OCA, OAB and ABC are equal.

MASTERJEE CONCEPTS

- The circumcentre, centroid and orthocentre are collinear.
- In any right angled triangle, the orthocentre coincides with the vertex containing the right angle.
- The mid-point of the hypotenuse of a right angled triangle is equidistant from the three vertices of a triangle.
- The mid-point of the hypotenuse of a right angled triangle is the circumcentreof the triangle.
- The centroid of the triangle lies on the line joining the circumcentre to the orthocentre and divides it in the ratio 1:2

Vaibhav Krishnan (JEE 2009,AIR 22)

9. PEDAL TRIANGLE

The triangle formed by the feet of the altitudes on the side of a triangle is called a pedal triangle.

In an acute angled triangle, orthocentre of ΔABC is the in-centre of the pedal triangle DEF.

Proof: Points F,H,D and B are concyclic

$\Rightarrow \angle FDH = \angle FBH = \angle ABE = \frac{\pi}{2} - A$

Similarly, points D, H, E and C are concyclic

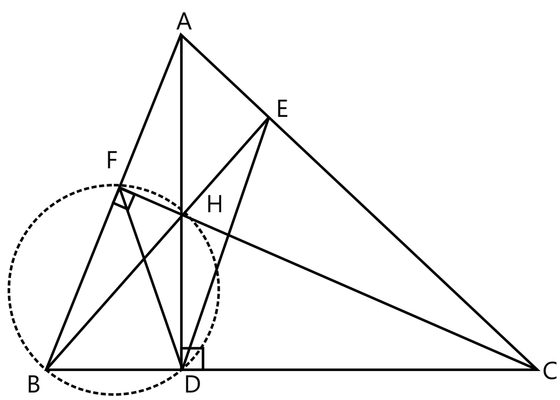

Figure 19.15

$\Rightarrow \angle HDE = \angle HCE = \angle ACF = \frac{\pi}{2} - A$

Thus, $\angle FDH = \angle HDE \Rightarrow AD$ is the angle bisector of $\angle FDE$. Hence, altitudes of ΔABC are internal angle bisectors of the pedal triangle. Thus, the orthocentre of ΔABC is the incentre of the pedal triangle DEF.

Sides of pedal triangle in acute angled triangle

In $\Delta AEF, AF = b\cos A, AE = c\cos A$

By cosine rule, $EF^2 = AE^2 + AF^2 - 2AE \times AF\cos(\angle EAF)$

$\Rightarrow EF^2 = b^2\cos^2 A + c^2\cos^2 A - 2bc\cos^3 A$

$\Rightarrow EF^2 = \cos^2 A\left(b^2 + c^2 - 2bc\cos A\right) = \cos^2 A\left(a^2\right) \quad \Rightarrow EF = a\cos A$

Circumradius of pedal triangle

Let circumradius be $R' \Rightarrow 2R' = \frac{EF}{\sin(\angle EDF)} = \frac{a\cos A}{\sin(\pi - 2A)} = \frac{a\cos A}{2\sin A\cos A} = \frac{a}{2\sin A} = R \Rightarrow R' = R/2$

MASTERJEE CONCEPTS

- The circle circumscribing the pedal triangle of a given triangle bisects the sides of the given triangle and also the lines joining the vertices of the given triangle to the orthocenter of the given triangle. This circle is known as "Nine point circle".
- The circumcentre of the pedal triangle of a given triangle bisects the line joining the circumcentre of the triangle to the orthocentre.
- It also passes through midpoint of the line segment from each vertex to the orthocenter.
- Orthocenter of triangle is in centre of pedal triangle.

Shrikant Nagori (JEE 2009, AIR 30)

10. ESCRIBED CIRCLES OF THE TRIANGLE

The circle which touches the sides BC and two sides AB and AC produced of a triangle ABC is called the escribed circle opposite to the angle A. Its radius is denoted by r_1. Similarly, r_2 and r_3 denote the radii of the escribed circles opposite to the angles B and C respectively. The centres of the escribed circles are called the ex-centres. The centre of the escribed circle opposite to the angle A is the point of intersection of the external bisector of angles B and C. The internal bisector also passes through the same point. This centre is generally denoted by I_1.

Formulae for r_1, r_2, r_3

In any ΔABC, we have (i) $r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}$

(ii) $r_1 = s\tan\frac{A}{2},\ r_2 = s\tan\frac{B}{2},\ r_3 = s\tan\frac{C}{2}$

(iii) $r_1 = a\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}},\ r_2 = b\frac{\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}},\ r_3 = c\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

(iv) $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},\ r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2},\ r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

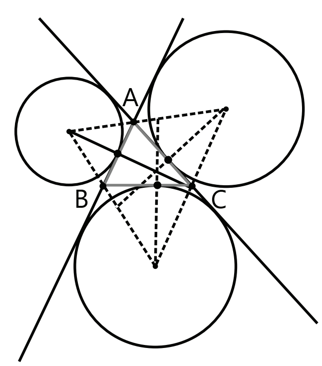

Figure 19.16

MASTERJEE CONCEPTS

If I_1 is the centre of the escribed circle opposite to the angle B, then

$$OI_1 = R\sqrt{1+8\sin\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}};\ OI_2 = R\sqrt{1+8\cos\frac{A}{2}.\sin\frac{B}{2}.\cos\frac{C}{2}};\ OI_3 = R\sqrt{1+8\cos\frac{A}{2}.\cos\frac{B}{2}.\sin\frac{C}{2}}$$

Where R is circum radius

- The Sum of the opposite angles of a cyclic quadrilateral is 180°.
- In a cyclic quadrilateral, the sum of the products of the opposites is equal to the product of diagonals. This is known as Ptolemy's theorem.
- If the sum of the opposite sides of a quadrilateral is equal, then and only then a circle can be inscribed in the quadrilateral.
- If I_1, I_2 and I_3 are the centres of escribed circles which are opposite to A,B and C respectively and I is the centre of the incircle, then triangle ABC is the pedal triangle of the triangle $I_1I_2I_3$ and I is the orthocenter of triangle $I_1I_2I_3$.
- The circle circumscribing the pedal triangle of a given triangle bisects the sides of the given triangle and also the lines joining the vertices of the given triangle to the orthocenter of the given triangle. This circle is also known as nine point circle.
- The circumradius of a cyclic quadrilateral, $R = \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}}$

Nitish Jhawar (JEE 2009, AIR 7)

11. LENGTH OF ANGLE BISECTOR AND MEDIANS

If m_a and β_a are the lengths of a median and an angle bisector from the angle A then,

$m_a = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$ and $\beta_a = \frac{2bc\cos\frac{A}{2}}{b+c}$. Note that $m_a^2+m_b^2+m_c^2 = \frac{3}{4}\left(a^2+b^2+c^2\right)$

Illustration 20: The ratio of the circumradius and in radius of an equilateral triangle is_______ **(JEE MAIN)**

Sol: Here, as we know, all angles of an equilateral triangle are 60^0, therefore by using formula of Circumradius and In radius we can obtain the required ratio.

$\frac{r}{R} = \frac{a\cos A+b\cos B+c\cos C}{a+b+c}$. In an equilateral triangle, $60^0 = A = B = C = \frac{(a+b+c)\cos 60^0}{(a+b+c)} = \frac{1}{2}$

Illustration 21: In a ΔABC, a=18 and b=24cm and c=30cm then find the value of r_1, r_2 and r_3. **(JEE MAIN)**

Sol: As we know, $r_1 = \frac{\Delta}{s-a}$, $r_2 = \frac{\Delta}{s-b}$ and $r_3 = \frac{\Delta}{s-c}$. Hence, we can solve the above problem by using this formula.

a=18cm, b=24cm,c=30cm; $\therefore 2s = a+b+c = 72$cm; s=36cm But, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\Delta = 216$ sq. units Then, $r_1 = \frac{\Delta}{s-a} = \frac{216}{18} = 12$cm; $r_2 = \frac{\Delta}{s-b} = \frac{216}{12} = 18$cm; $r_3 = \frac{\Delta}{s-c} = \frac{216}{6} = 36$cm

So, r_1, r_2, r_3 are 12cm, 18cm, and 36cm respectively.

Illustration 22: If the exradii of a triangle are in HP, the corresponding sides are in ____ **(JEE MAIN)**

Sol: Here, in this problem, r_1, r_2 and r_3 are in H.P.

$$\Rightarrow \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3} \text{ are in A.P.} \Rightarrow \frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta} \text{ are in A.P.} \Rightarrow s-a, s-b, s-c \text{ are in A.P.}$$

$$\Rightarrow -a, -b, -c \text{ are in A.P.} \Rightarrow a, b, c \text{ are in A.P.}$$

Illustration 23: Find the value of $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3}$. **(JEE ADVANCED)**

Sol: By using $r_1 = \frac{\Delta}{s-a}$, $r_2 = \frac{\Delta}{s-b}$ and $r_3 = \frac{\Delta}{s-c}$, we can solve the above problem.

$$\frac{(b-c)}{r_1} + \frac{(c-a)}{r_2} + \frac{(a-b)}{r_3} = (b-c)\left(\frac{s-a}{\Delta}\right) + (c-a)\left(\frac{s-b}{\Delta}\right) + (a-b)\left(\frac{s-c}{\Delta}\right)$$

$$= \frac{(s-a)(b-c) + (s-b)(c-a) + (s-c)(a-b)}{\Delta}$$

$$= \frac{s(b-c+c-a+a-b) - [ab-ac+bc-ba+ac-bc]}{\Delta} = \frac{0}{\Delta} = 0$$

Illustration 24: Find the value of the $r\cot\frac{B}{2}\cot\frac{C}{2}$. **(JEE ADVANCED)**

Sol: Here, in this problem, r=4RsinA/2.sinB/2.sinC/2. By putting this value, we can solve the above problem.

$$r\cot B/2.\cot C/2 = 4R\sin A/2\sin B/2.\sin C/2.\frac{\cos B/2}{\sin B/2}.\frac{\cos C/2}{\sin C/2} \quad [\text{as } r=4R\sin A/2.\sin B/2.\sin C/2]$$

$$= 4R.\sin A/2.\cos B/2.\cos C/2 = r_1 \{\text{as}, r_1 = 4R\sin A/2.\cos B/2.\cos C/2\}$$

$$\therefore r\cot B/2.\cot C/2 = r_1$$

12. EXCENTRAL TRIANGLE

The triangle formed by joining the three excentres I_1, I_2 and I_3 of ΔABC is called the excentral or excentric triangle. Note that:

(i) The incentre I of ΔABC is the orthocentre of the excentral $\Delta I_1I_2I_3$.

(ii) ΔABC is the pedal triangle of the $\Delta I_1I_2I_3$.

(iii) The sides of the excentral triangle are

$4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$ and $4R\cos\frac{C}{2}$ and

Its angles are $\frac{\pi}{2}-\frac{A}{2}, \frac{\pi}{2}-\frac{B}{2}$ and $\frac{\pi}{2}-\frac{C}{2}$.

(iv) Distance between the incentre and excentre

$II_1 = 4R\sin\frac{A}{2}$; $II_2 = 4R\sin\frac{B}{2}$; $II_3 = 4R\sin\frac{C}{2}$.

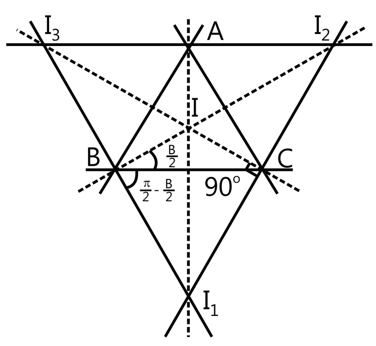

Figure 19.17

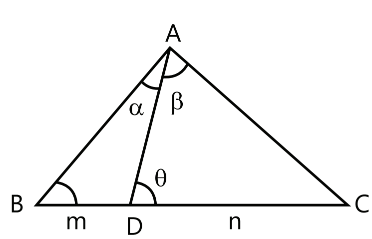

Figure 19.18

13. M-N THEOREM (RATIO FORMULA)

If D be a point on the side BC of a ΔABC such that BD:DC = m:n and $\angle ADC = \theta$, $\angle BAC = \alpha$ and $\angle DAC = \beta$.

(a) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$ (b) $(m+n)\cot\theta = n\cot B - m\cot C$

Proof: (a) Given that, $\frac{BD}{DC} = \frac{m}{n}$ and $\angle ADC = \theta$

$\because \angle ADB = (180^o - \theta)$; $\angle BAD = \alpha$ and $\angle DAC = \beta$

$\therefore \angle ABD = 180^o - (\alpha + 180^o - \theta) = \theta - \alpha$ and $\angle ACD = 180^o - (\theta + \beta)$

From $\Delta ABD, \frac{BD}{\sin\alpha} = \frac{AD}{\sin(\theta - \alpha)}$ ... (i)

From $\Delta ADC, \frac{DC}{\sin\beta} = \frac{AD}{\sin[180^o - (\theta+\beta)]}$ or $\frac{DC}{\sin\beta} = \frac{AD}{\sin(\theta+\beta)}$ ... (ii)

dividing (i) by (ii), then $\frac{BD\sin\beta}{DC\sin\alpha} = \frac{\sin(\theta+\beta)}{\sin(\theta-\alpha)}$ or $\frac{m}{n}\frac{\sin\beta}{\sin\alpha} = \frac{\sin\theta.\cos\beta + \cos\theta.\sin\beta}{\sin\theta.\cos\alpha - \cos\theta\sin\alpha}$

or $m\sin\theta\sin\beta\cos\alpha - m\cos\theta.\sin\alpha.\sin\beta = n\sin\alpha\sin\theta\cos\beta + n\sin\alpha\cos\theta\sin\beta$

$m\cot\alpha - m\cot\theta = n\cot\beta + n\cot\theta$ [dividing both sides by $\sin\alpha\sin\beta\sin\theta$] or $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(b) Given $\frac{BD}{DC} = \frac{m}{n}$ and $\angle ADC = \theta$; $\therefore \angle ADB = 180^o - \theta$; $\angle ABD = B$ and $\angle ACD = C$

and $\angle BAD = 180^o - (180^o - \theta + B) = \theta - B$; $\therefore \angle DAC = 180^o - (\theta + C)$

and now from $\Delta ABD. \frac{BD}{\sin(\theta - B)} = \frac{AD}{\sin B}$ ... (i)

and from $\Delta ADC, \frac{DC}{\sin[180^o - (\theta + C)]} = \frac{AD}{\sin C}$ or $\frac{DC}{\sin(\theta + C)} = \frac{AD}{\sin C}$ ... (ii)

dividing (i) by (ii), then $\frac{BD}{DC}.\frac{\sin(\theta+C)}{\sin(\theta-B)} = \frac{\sin C}{\sin B}$ or, $\frac{m}{n}\frac{\sin\theta\cos C + \cos\theta\sin C}{\sin\theta\cos B - \cos\theta\sin B} = \frac{\sin C}{\sin B}$

or, $m\sin\theta\cos C\sin B + m\cos\theta\sin C\sin B = n\sin\theta\sin C\cos B - n\cos\theta\sin B\sin C$

or, $m\cot C + m\cot\theta = n\cot B - n\cot\theta$ [dividing both sides by $\sin B\sin C\sin\theta$]

or, $(m+n)\cot\theta = n\cot B - m\cot C$

Illustration 25: In a triangle ABC, if $\cot\frac{A}{2}\cot\frac{B}{2} = c$, $\cot\frac{B}{2}\cot\frac{C}{2} = a$ and $\cot\frac{C}{2}\cot\frac{A}{2} = b$,

then find the value of $\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}$. **(JEE MAIN)**

Sol: Here, by using trigonometric ratios of half angle, we can solve above problem.

$\cot\frac{A}{2}\cot\frac{B}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)} \times \frac{s(s-b)}{(s-c)(s-a)}} = c$; $\frac{s}{s-c} = c \Rightarrow \frac{1}{s-c} = \frac{c}{s}$

Similarly $\frac{1}{s-a} = \frac{a}{s}$ and $\frac{1}{s-b} = \frac{b}{s}$

So that $\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} = \frac{a+b+c}{s} = \frac{2s}{s} = 2$

14. SOLUTION OF DIFFERENT TYPES OF TRIANGLE

In a triangle, there are six elements- three sides and three angles. In plane geometry, we have done that if three of the elements are given, at least one of which must be side, then the other three elements can be uniquely determined. The procedure of determining unknown elements from the known elements is called solving a triangle.

Solution of a right angled triangle

Case I: When two sides are given: Let the triangle be right angled at C. Then we can determine the remaining elements as given in the table.

	Given	Required
(i)	a, b	$\tan A = \frac{a}{b}, B = 90^o - A, c = \frac{a}{\sin A}$
(ii)	a, c	$\sin A = \frac{a}{c},\ b = c\cos A,\ B = 90^o - A$

Case II: When a side and an acute angle are given: In this case, we can determine the remaining elements as given in the table.

	Given	Required
(i)	a, A	$B = 90^o - A, b = a\cot A, c = \frac{a}{\sin A}$
(ii)	c, A	$B = 90^o - A, a = c\sin A, b = c\cos A$

Solution of a triangle in general

Case I: When three sides a, b, c are given: In this case, the remaining elements are determined by using the following formulae. $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where 2s = a + b + c

$$\sin A = \frac{2\Delta}{bc}, \sin B = \frac{2\Delta}{ac}, \sin C = \frac{2\Delta}{ab}. \text{ OR } \tan\left(\frac{A}{2}\right) = \frac{\Delta}{s(s-a)}, \tan\left(\frac{B}{2}\right) = \frac{\Delta}{s(s-b)}, \tan\left(\frac{c}{2}\right) = \frac{\Delta}{s(s-c)}$$

Case II: When two sides a, b and the included angle C are given: In this case, we use the following formulae:

$$\Delta = \frac{1}{2}ab\sin C,\ \tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\left(\frac{2C}{2}\right);\ \frac{A+B}{2} = 90^o - \frac{C}{2} \text{ and } c = \frac{a\sin C}{\sin A}$$

Case III: When one side a and two angle A and B are given: In this case, we use the following formulae to determine the remaining elements.

$$A + B + C = 180^o;\ C = 180^o - (A+B) \text{ and } c = \frac{a\sin C}{\sin A};\ \Delta = \frac{1}{2}ca\sin B$$

Case IV: When two sides a, b and the A opposite to one side is given: In this case, we use the following formulae.

$$\sin B = \frac{b}{a}\sin A \quad \text{... (i)}$$

$$C = 180^o - (A+B),\ c = \frac{a\sin C}{\sin A}$$

From (i), the following possibilities will arise:

When A is an acute angle and $a < b\sin A$.

In this case, the relation $\sin B = \frac{b}{a}\sin A$ gives that $\sin B > 1$, which is impossible. Hence no triangle is possible.

When A is an acute angle and $a = b\sin A$.

In this case, only one triangle is possible which is right angled at B.

When A is an acute angle and $a > b\sin A$

In this case, there are two values of B given by $\sin B = \frac{b\sin A}{a}$ say B_1 and B_2 such that $B_1 + B_2 = 180^o$ and side c can be obtained by using $c = \frac{a\sin C}{\sin A}$

Some useful results:

Solution of oblique triangles:

The triangle which are not right angled are known as oblique triangles. The problems on solving an oblique triangle lie in the following categories:

(a) When three sides are given

(b) When two side and included angle are given

(c) When one side and two angles are given

(d) When all the three angles are given

(e) Ambiguous case in solution of triangle

When the three sides are given: When three sides a, b, c of a triangle are given, then to solve it, we have to find its three angles A,B,C. For this cosine rule can be used.

When two sides and included angle are given: Problem based on finding the angles when any two sides and the angles between them or any two sides and the difference of the opposite angles to them are given, Napier's analogy can be used.

When one side and two angles are given: Problems based on finding the sides and angles when any two and side opposite to one of them are given, then sine rule can be used.

When all the three angles are given: In this case unique solution of triangle is not possible. In this case only the ratio of the sides can be determined.

For this the formula, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ can be used

Ambiguous case in solution of triangles: When any two sides and one of the corresponding angles are given, under certain additional conditions, two triangles are possible. The case when two triangles are possible is called the ambiguous case.

In fact, when any two sides and the angle opposite to one of them are given either no triangle is possible or only one triangle is possible or two triangles are possible.

Now, we will discuss the case when two triangles are possible.

Illustration 26: Solve the triangle, if $b = 72.95$, $c = 82.31$, $B = 42^o47'$ **(JEE MAIN)**

Sol: By using sine rule i.e. $\frac{\sin C}{c} = \frac{\sin B}{b}$, we can solve the given triangle.

(i) To find C $\frac{\sin C}{c} = \frac{\sin B}{b} \Rightarrow \sin C = \frac{c\sin B}{b} = \frac{82.31 \times \sin 42^o 47'}{72.95} = 0.7663$

$C = \sin^{-1}(0.7663)$ $C_1 = 50^o 1' 12''$ and $C_2 = 129^o 58' 48''$

I solution	II solution
$C = 50^o1'12''$ $A = 180^o - (B+C)$ $A = 180^o - (42^o47' + 50^o1'12'') = 87^o11'48''$ To find a $\frac{a}{\sin A} = \frac{b}{\sin B}$ $a = \frac{b\sin A}{\sin B} = \frac{72.95 \times \sin 87^o11'48''}{\sin 42^o27'}$ $a = 107.95$	$C = 129^o58'48''$ $A = 180^o - (B+C)$ $= 180^o - (42^o47' + 129^o58'48'') = 7^o14'12''$ To find a $\frac{a}{\sin A} = \frac{b}{\sin B}$ $a = \frac{b\sin A}{\sin B} = \frac{72.95 \times \sin 7^o14'12''}{\sin 42^o27'}$ $a = 13.62$

$\therefore$ Two solutions are

$C_1 = 50^o1'12''$ $A_1 = 87^o11'48''$ $a_1 = 107.95$ $C_2 = 129^o58'48''$ $A_2 = 7^o14'12''$ $a_2 = 13.62$

Geometrically, we draw the triangle with given data c,b and angle B.

(a) If $AN(= c\sin B) = b$ (exactly). The triangle is a right angled triangle.

(b) If $AN(= c\sin B) > b$, the triangle cannot be drawn.

(c) If $AN(= c\sin B) < b < c$, two triangles are possible.

(d) b>c, only one triangle is possible.

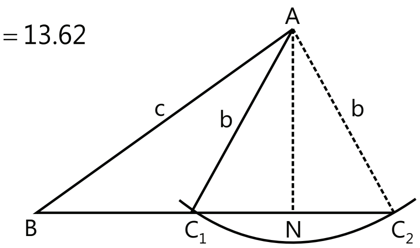

Figure 19.19

Illustration 27: In a triangle ABC, b=16cm, c=25cm, and $B = 33^o15'$. Find the angle C. **(JEE MAIN)**

Sol: Simply by using sine rule, we can find out the angle C.

We know that, $\frac{\sin C}{c} = \frac{\sin B}{b}$ [Here, b=16cm, c=25cm, B= $33^o15'$]

$\sin C = \frac{c}{b}\sin B = \frac{25\sin 33^o15'}{16} = 0.8567$; $C = \sin^{-1}(0.8567) = 58^o57'$; $C_1 = 58^o57'$; $C_2 = 180^o - 58^o57' = 121^o3'$

PROBLEM SOLVING TACTICS

In the application of sine rule, the following points are to be noted. We are given one side a and some other side x is to be found. Both these are in different triangles. We choose a common side y of these triangles. Then apply sine rule for a and y in one triangle and for x and y for the other triangle and eliminate y. Thus, we will get the unknown side x in terms of a.

In the adjoining figure, a is the known side of ΔABC and x is the unknown side of triangle ACD. The common side of these triangles is AC=y(say). Now, apply sine rule.

$\therefore \frac{a}{\sin\alpha} = \frac{y}{\sin\beta}$ ...(i) and $\frac{x}{\sin\theta} = \frac{y}{\sin\gamma}$ ... (ii)

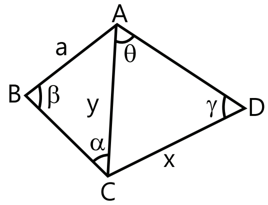

Figure 19.20

Dividing (ii) by (i) we get, $\frac{x\sin\alpha}{a\sin\theta} = \frac{\sin\beta}{\sin\gamma}$; $\therefore x = \frac{a\sin\beta\sin\theta}{\sin\alpha\sin\gamma}$

In case of generalized triangle problems, option verification is very useful using equilateral, isosceles or right angle triangle properties. So, it is advised to remember properties of these triangles.

FORMULAE SHEET

(a) In ΔABC, $\angle A + \angle B + \angle C = \pi$

(a) $\sin(B+C) = \sin(\pi - A) = \sin A$

(b) $\cos(C+A) = \cos(\pi - B) = -\cos B$

(c) $\sin\frac{A+B}{2} = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cos\frac{C}{2}$

(d) $\cos\frac{B+C}{2} = \cos\left(\frac{\pi}{2} - \frac{A}{2}\right) = \sin\frac{A}{2}$

(b) **Sine rule:** In, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ Where R = Circumradius and a, b, c are sides of triangle.

(c) **Cosine rule:** $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$, $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

(d) **Trigonometric ratios of half – angles:**

(a) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ where 2s = a + b + c; **(b)** $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$; **(c)** $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

(e) **Area of a triangle :** $\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$

(f) **Heron's formula :** $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$.

(g) **Circumcircle Radius :** $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$

(h) **Incircle Radius :** **(a)** $r = \frac{\Delta}{s}$; **(b)** $r = (s-a)\tan\left(\frac{A}{2}\right)$, $r = (s-b)\tan\left(\frac{B}{2}\right)$ and $r = (s-c)\tan\left(\frac{C}{2}\right)$

(i) **Radius of the Escribed Circle :**

(a) $r_1 = \frac{\Delta}{s-a}$, $r_2 = \frac{\Delta}{s-b}$, $r_3 = \frac{\Delta}{s-c}$

(b) $r_1 = s\tan\frac{A}{2}$, $r_2 = s\tan\frac{B}{2}$, $r_3 = s\tan\frac{C}{2}$

(c) $r_1 = a\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$, $r_2 = b\frac{\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}$, $r_3 = c\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

(d) $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$, $r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$, $r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

(j) **Length of Angle bisector and Median:**

$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$ and $\beta_a = \frac{2bc\cos\frac{A}{2}}{b+c} \Rightarrow m_a$ - length of median, β_a - length of bisector.