3. SEQUENCES AND SERIES

1. SEQUENCE

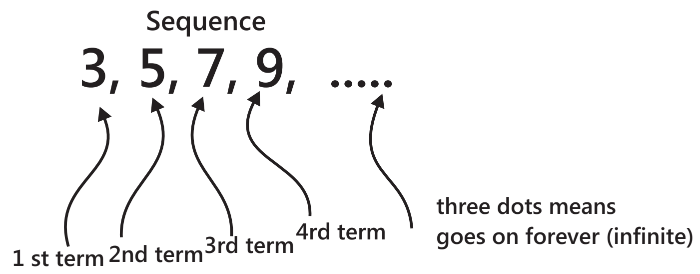
1.1 Introduction

A sequence can be defined as an ordered collection of things (usually numbers) or a set of numbers arranged one after another. Sometimes, sequence is also referred as progression. The numbers a_1, a_2, a_3 .....a_n are known as terms or elements of the sequence. The subscript is the set of positive integers 1, 2, 3..... that indicates the position of the term in the sequence. T_n is used to denote the n^{th} term.

Some examples of a sequence are as follows:

0, 7, 26......................., 1, 4, 7, 10......................., 2, 4, 6, 8.......................

Note: The minimum number terms in a sequence should be 3.



("term"", element" or ‘‘ member"" mean the same thing)

Figure 3.1

1.2 Finite and Infinite Sequences

A sequence containing a finite number of terms is called a finite sequence. If the sequence contains a infinite number of terms, it is known as an infinite sequence. It is infinite in the sense that it never ends. Examples of infinite and finite sequences are as follows:

{1, 2, 3, 4......} is an infinite sequence

{20, 25, 30, 35....} is an infinite sequence

{1, 3, 5, 7} is the sequence of the first 4 odd numbers, which is a finite sequence

1.3 Rule

A sequence usually has a rule, on the basis of which the terms in the sequence are built up. With the help of this rule, we can find any term involved in the sequence. For example, the sequence {3, 5, 7, 9} starts at the number 3 and jumps 2 every time.

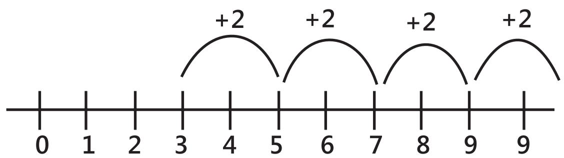


Figure 3.2

As a Formula:

Saying 'start at the number 3 and jump 2 every time' is fine, but it does not help to calculate the 10^{th} term or 100^{th} term or n^{th} term. Hence, we want a formula for the sequence with "n" in it (where n is any term number). What would the rule for {3, 5, 7, 9.......} be? First, we can see the sequence goes up 2 every time; hence, we can guess that the rule will be something like '2 times n' (where 'n' is the term number). Let us test it out.

n	Test Rule	Term
1	$2n = 2 \times 1 = 2$	3
2	$2n = 2 \times 2 = 4$	5
3	$2n = 2 \times 3 = 6$	7

That nearly worked! But it is less by 1 every time. Let us try changing it to 2n+1.

n	Test Rule	Term
1	$2n + 1 = 2 \times 1 + 1 = 3$	3
2	$2n + 1 = 2 \times 2 + 1 = 5$	5
3	$2n + 1 = 2 \times 3 + 1 = 7$	7

That Works: Therefore, instead of saying 'starts at the number 3 and jumps 2 every time,' we write the expression 2n + 1. We can now calculate, e.g. the 100^{th} term as $2 \times 100 + 1 = 201$.

1.4 Notation

The notation T_n is used to represent the general term of the sequence. Here, the position of the term in the sequence is represented by n. To mention for the '$5'^{th}$ term, just write T_5.

Thus, the rule for {3, 5, 7, 9...} can be written as the following equation: $T_n = 2n + 1$.

To calculate the 10^{th} term, we can write $T_{10} = 2n + 1 = 2 \times 10 + 1 = 21$

Illustration 1: Find out the first 4 terms of the sequence, $\{T_n\} = \{-1/n\}^n$. **(JEE MAIN)**

Sol: By substituting n = 1, 2, 3 and 4 in$\{T_n\} = \{-1/n\}^n$, we will get the first 4 terms of given sequence.

$T_1 = (-1/1)^1 = -1$

$T_2 = (-1/2)^2 = 1/4$

$T_3 = (-1/3)^3 = -1/27$

$T_4 = (-1/4)^4 = 1/256$

$\Rightarrow \{T_n\} = \{-1, ¼, -1/27, 1/256 ...\}$

Illustration 2: Write the sequence whose n^{th} term is (i) 2^n and (ii) $\log(nx)$. **(JEE MAIN)**

Sol: By substituting n = 1, 2, 3........, we will get the sequence.

(i) n^{th} term $= 2^n$

$a_1 = 2^1, a_2 = 2^2$.......

Sequence $\Rightarrow 2^1, 2^2, \ldots\ldots 2^n$

(ii) n^{th} term $= a_n = \log(nx)$

$a_1 = \log(x)$

$a_2 = \log(2x), a_n = \log(nx)$

Sequence $\Rightarrow \log(x), \log(2x), \ldots\ldots\ldots \log(nx)$

2. SERIES

Series is something that we get from a given sequence by adding all the terms. If we have a sequence as $T_1, T_2, \ldots, T_n$, then the series that we get from this sequence is $T_1 + T_2 + \ldots + T_n$. S_n is used to represent the sum of n terms. Hence, $S_n = T_1 + T_2 + \ldots + T_n$

3. SIGMA AND PI NOTATIONS

3.1 Sigma Notation

The meaning of the symbol Σ (sigma) is summation. To find the sum of any sequence, the symbol Σ (sigma) is used before its n^{th} term. For example:

(i) $\sum_{n=1}^{9} n = 1+2+3+\ldots\ldots+9$

(ii) $\sum_{r=1}^{n} r^a$ or $\sum n^a = 1^a+2^a+3^a+\ldots\ldots+n^a$

(iii) $\sum_{i=1}^{5} \frac{i+1}{2i+4} = \frac{1+1}{2\times1+4} + \frac{2+1}{2\times2+4} + \frac{3+1}{2\times3+4} + \frac{4+1}{2\times4+4} + \frac{5+1}{2\times5+4}$

Properties of Σ (Sigma)

(i) $\sum_{i=1}^{k} a = a + a + a\ldots$ (k times) $= ka$, where a is a constant.

(ii) $\sum_{i=1}^{k} ai = a\sum_{i=1}^{k} i$, where a is a constant.

(iii) $\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$

(iv) $\sum_{i=i_0}^{i_n} \sum_{j=j_0}^{j_n} a_i a_j = \sum_{j=j_0}^{j_n} \sum_{i=i_0}^{i_n} a_i a_j$

3.2 Pi Notation

The symbol $\prod$ denotes the product of similar terms. For example:

(i) $\prod_{n=1}^{6} n = 1 \times 2 \times 3 \times 4 \times 5 \times 6$

(ii) $\prod_{n=1}^{k} n^m = 1^m \times 2^m \times 3^m \times 4^m \times \ldots\ldots\ldots\ldots \times k^m$

(iii) $\prod_{n=1}^{k} n = 1 \times 2 \times 3 \times \ldots\ldots \times k = k!$

4. ARITHMETIC PROGRESSION

The sequence in which the successive terms maintain a constant difference is known as an arithmetic progression (AP). Consider the following sequences:

$$\begin{array}{cccc} a, & a+d, & a+2d, & a+3d \\ T_1, & T_2, & T_3, & T_4 \end{array}$$

$\because\ T_2 - T_1 = T_3 - T_2 = T_4 - T_3 =$ constant (common difference)

The given sequence is an example of AP. The set of natural numbers is also an example of AP.

4.1 General Term

General term (n^{th} term) of an AP is given by $T_n = a + (n - 1)\,d$, where a is the first term of the sequence and d is the common difference of the sequence.

Note:

(i) General term is also denoted by ℓ (last term).

(ii) n (number of terms) always belongs to the set of natural numbers.

(iii) Common difference can be zero, + ve or – ve.

If $d > 0 \Rightarrow$ increasing AP and the sequence tends to $+\infty$

If $d < 0 \Rightarrow$ decreasing AP and the sequence tends to $-\infty$

If $d = 0 \Rightarrow$ constant AP (all the terms remain same)

(iv) The n^{th} term from end is $(m - n + 1)$ term from the beginning, where m is the total number of terms and is given by the following expression:

$T_{m-n+1} = T_m - (n-1)\,d$

MASTERJEE CONCEPTS

- If the m^{th} term is n and the n^{th} term is m, then the $(m + n)^{th}$ term is 0.
- If m times the m^{th} term is equal to n times the n^{th} term, then the $(m + n)^{th}$ term is 0.

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Illustration 3: If the 5^{th} term of an AP is 17 and its 7th term is 15, then find the 22^{th} term. **(JEE MAIN)**

Sol: Using the formula $T_n = a + (n - 1)\,d$, we can solve above problem.

Given $a + 4d = 17$ and $a + 6d = 15$

$\Rightarrow\quad 2d = -2 \Rightarrow d = -1,\ a = 21$

$\therefore\quad T_{22} = 21 - 21 = 0$

Illustration 4: If 11 times the 11^{th} term of an AP is equal to 9 times the 9^{th} term, then find the 20^{th} term. **(JEE MAIN)**

Sol: By solving $11\,(a + 10d) = 9\,(a + 8d)$, we will get the value of a and d.

$\therefore\ 2a = -38\,d \Rightarrow a = -19d$

$\therefore\ 20^{th}$ term $= a + 19d = 0$

Illustration 5: Check whether the sequences given below are AP or not. **(JEE MAIN)**

(i) $T_n = n^2$ (ii) $T_n = an + b$

Sol: By taking the difference of two consecutive terms, we can check whether the sequences are in AP or not.

(i) $T_n = n^2;\ T_{n-1} = (n - 1)^2$

Difference = $T_n - T_{n-1} = n^2 - (n-1)^2 = n^2 - (n^2 - 2n + 1) = 2n - 1$

This difference varies with respect to the term. Hence, the sequence is not an AP.

(ii) $T_n = an + b$; $T_{n-1} = a(n-1) + b$

Difference = $(an + b) - (a(n-1) + b) = a$ (constant)

Hence, the sequence is an AP.

Illustration 6: The 2nd, 31st and the last term of an AP are given as $7\frac{3}{4}, \frac{1}{2}$ and $-6\frac{1}{2}$, respectively. Find the first term and the number of terms. **(JEE MAIN)**

Sol: Using $T_n = a + (n-1)d$, we can get the first term and common difference. Suppose a be the first term and d be the common difference of the AP.

Given, $T_2 = 7\frac{3}{4} \Rightarrow a + d = \frac{31}{4}$ .... (i)

$T_{31} = \frac{1}{2} \Rightarrow a + 30d = \frac{1}{2}$ .... (ii)

Subtracting (i) from (ii), we get $29d = \frac{1}{2} - \frac{31}{4} = -\frac{29}{4} \Rightarrow d = \frac{-1}{4}$

Putting the value of d in (i), we get $a - \frac{1}{4} = \frac{31}{4} \Rightarrow a = \frac{31}{4} + \frac{1}{4} = \frac{32}{4} = 8$

Suppose the number of terms be n, so that $T_n = -\frac{13}{2}$

i.e. $a + (n-1)d = -\frac{13}{2} \Rightarrow 8 + (n-1)\left(-\frac{1}{4}\right) = -\frac{13}{2}$

$\Rightarrow 32 - n + 1 = -26 \Rightarrow n = 59$

Hence, the first term = 8 and the number of terms = 59.

Illustration 7: Prove that the square roots of three unequal prime numbers cannot be three terms of an AP. **(JEE ADVANCED)**

Sol: Here by Considering $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ to be the λ^{th}, μ^{th} and v^{th} terms of an AP and solving them using $T_n = a + (n-1)d$, we prove the problem.

If possible let $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ be the three terms of an AP. a, a +d, a + 2d........., where $p \neq q \neq r$ and they are prime numbers. Let them be the λ^{th}, μ^{th} and v^{th} terms, respectively.

$\therefore \sqrt{p} = a + (\lambda - 1)d$

$\sqrt{q} = a + (\mu - 1)d$

$\sqrt{r} = a + (\nu - 1)d$

$\therefore \sqrt{p} - \sqrt{r} = (\lambda - \mu)d$

Also, $\sqrt{q} - \sqrt{r} = (\mu - v)d$

$$\therefore \frac{\sqrt{p}-\sqrt{q}}{\sqrt{q}-\sqrt{r}} = \frac{\lambda-\mu}{\mu-v} \text{ or } \frac{\left(\sqrt{p}-\sqrt{q}\right)\left(\sqrt{q}+\sqrt{r}\right)}{\left(\sqrt{q}-\sqrt{r}\right)\left(\sqrt{q}+\sqrt{r}\right)} = \frac{\lambda-\mu}{\mu-v}$$

or $\sqrt{pq} + \sqrt{pr} - q - \sqrt{qr} = \frac{\lambda-\mu}{\mu-v}(q-r)$ or $\sqrt{pq} + \sqrt{pr} - \sqrt{qr} = q + \frac{\lambda-\mu}{\mu-v}(q-r)$ = rational number

Since p, q, r are unequal primes, $\sqrt{pq}, \sqrt{pr}$ and $\sqrt{qr}$ are unequal pure irrational numbers. Thus, LHS is irrational, but irrational $\neq$ rational.

Hence, the problem is proved.

Illustration 8: If x, y and z are real numbers satisfying the equation $25(9x^2 + y^2) + 9z^2 - 15(5xy + yz + 3zx) = 0$, then prove that x, y and z are in AP. **(JEE ADVANCED)**

Sol: By solving the equation $25(9x^2 + y^2) + 9z^2 - 15(5xy + yz + 3zx) = 0$, we can prove that x, y and z are in AP.

We have

$$(15x)^2 + (5y)^2 + (3z)^2 - (15x)(5y) - (5y)(3z) - (3z)(15x) = 0$$

$$\Rightarrow (15x - 5y)^2 + (5y - 3z)^2 + (3z - 15x)^2 = 0$$

$$\Rightarrow 15x - 5y = 0,\ 5y - 3z = 0,\ 3z - 15x = 0$$

$$\Rightarrow 15x = 5y = 3z \Rightarrow \frac{x}{1} = \frac{y}{3} = \frac{y}{5} \ (= k \text{ say})$$

$$\therefore x = k,\ y = 3k,\ z = 5k$$

Thus, x, y and z are in AP.

Illustration 9: Let $a_1, a_2, a_3, \ldots, a_n$ be in AP, where $a_1 = 0$ and the common difference $\neq 0$. Show that

$$\frac{a_3}{a_2} + \frac{a_4}{a_3} + \frac{a_5}{a_4} + \ldots\ldots \frac{a_n}{a_{n-1}} - a_2\left(\frac{1}{a_2} + \frac{1}{a_3} + \ldots\ldots + \frac{1}{a_{n-2}}\right) = \frac{a_{n-1}}{a_2} + \frac{a_2}{a_{n-1}}$$ **(JEE ADVANCED)**

Sol: Given $a_1 = 0$ and $d = a_2 - a_1 = a_2 - 0 = a_2$

By solving LHS and RHS separately, we can solve the problem.

$$\text{LHS} = \frac{a_3 - a_2}{a_2} + \frac{a_4 - a_2}{a_3} + \frac{a_5 - a_2}{a_4} + \ldots + \frac{a_{n-1} - a_2}{a_{n-2}} + \frac{a_n}{a_{n-1}}$$

$$= \frac{a_2}{a_2} + \frac{a_3}{a_3} + \frac{a_4}{a_4} + \ldots\ldots + \frac{a_{n-2}}{a_{n-2}} + \frac{a_n}{a_{n-1}}$$

$$= (n-3) + \frac{a_n}{a_{n-1}} = (n-3) + \frac{a_1 + (n-1)d}{a_1 + (n-2)d} = (n-3) + \frac{n-1}{n-2} \quad \{\because a_1 = 0\}$$

$$\text{RHS} = \frac{a_1 + (n-2)d}{a_2} + \frac{a_2}{a_1 + (n-2)d} = (n-2) + \frac{1}{n-2} = (n-3) + 1 + \frac{1}{n-2} = (n-3) + \frac{n-1}{n-2}$$

$\therefore$ LHS = RHS

MASTERJEE CONCEPTS

A sequence obtained by multiplication or division of corresponding terms of two APs may not be in AP

For example, let the first AP be 2, 4, 6, 8,................ and the second AP be 1, 2, 3, 4, 5.......

Multiplying these two, we get 2,8,18,32, ..., which is clearly not an AP

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4.2 Series of an AP

Series of an AP can be obtained as

$S_n = a + (a + d) + (a + 2d) \ldots\ldots [a+(n-1)d]$

$S_n = [a + (n-1)d] + [a+(n-2)d] \ldots\ldots + a$ (writing in the reverse order)

$\therefore\ 2S_n = n(2a + (n-1)\,d)$

$\therefore$ Sum to n terms, $S_n = \frac{n}{2}(2a+(n-1)d) = \frac{n}{2}\left(T_1 + T_n\right)$

Illustration 10: Find the sum of the first 19 terms of an AP when $a_4 + a_8 + a_{12} + a_{16} = 224$. **(JEE MAIN)**

Sol: We need to find out the sum of the first 19 terms of an AP, i.e. $\frac{19}{2}$ (2a + 18d), and we can represent the given equation as (a + 3d) + (a + 7d) + (a + 11d) + (a + 15d) = 224.

Given (a + 3d) + (a + 7d) + (a + 11d) + (a +15d) = 224

$\Rightarrow\ 4a + 36d = 224 \Rightarrow a + 9d = 56$

Sum of the first 19 terms $\Rightarrow S = \frac{19}{2}\ (2a + 18d) = \frac{19}{2}\times 2\times 56 = 1064$

Illustration 11: The sum of n terms of two arithmetic progressions is in the ratio of $\frac{7n+1}{4n+27}$. Find ratio of the 11^{th} terms? **(JEE MAIN)**

Sol: Since we know the sum of n terms, i.e. $S_n = \frac{n}{2}\left[2a+\left(n-1\right)d\right]$, we can write the equation as

$\frac{7n+1}{4n+27} = \frac{\frac{n}{2}(2a_1+(n-1)d_1)}{\frac{n}{2}(2a_2+(n-1)d_2)}$. Hence, by putting n = 11 in this equation, we can obtain the ratio of the 11^{th} terms.

$$\frac{7n+1}{4n+27} = \frac{a_1+\left(\frac{n-1}{2}\right)d_1}{a_1+\left(\frac{n-1}{2}\right)d_2}$$

We want the ratio of $\frac{a_1+10d_1}{a_2+10d_2}$. Hence, $\frac{n-1}{2} = 10\ \Rightarrow n = 21$

$$\Rightarrow \frac{a_1+10d_2}{a_2+10d_2} = \frac{148}{111}$$

Illustration 12: In an AP of n terms, prove that the sum of the k^{th} term from the beginning and the k^{th} term from the end is independent of k and equal to the sum of the first and last terms. **(JEE MAIN)**

Sol: Using the formula $T_k = a+(k-1)d$ and $T_{n-k+1} = [a+(n-k)d]$, we can obtain the k^{th} term from the beginning and end, respectively, and after that by adding these values we can prove the given problem.

Suppose a be the first term and d be the common difference of the AP.

$\therefore$ k^{th} term from the beginning = $T_k = a+(k-1)d$ ... (i)

Let l be the last term of the AP and l = a + (n – 1) d

The k^{th} term from the end of the given AP is the $(n - k + 1)^{th}$ term from the beginning.

$\therefore \quad T_{n-k+1} = [a + (n-k)d]$ ...(ii)

Adding (i) and (ii), we get

$\therefore$ The required sum = $T_k + T_{n-k+1} = [a + (k-1)d] + \left[a + (n-k)d\right]$ 2a + (k – 1 + n – k) d

= 2a + (n – 1) d .... (iii), which is independent of k

Moreover, the sum of the first and last terms = a + l = a + [a + (n – 1) d] = 2a + (n – 1) d. ...(iv)

Thus, the sum of the first and last terms is independent of k and (3) = (4). Hence proved.

Illustration 13: If $a_1, a_2, a_3, \ldots, a_n$ is an AP of non-zero terms , then prove that **(JEE ADVANCED)**

$$\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \ldots.. + \frac{1}{a_{n-1}a_n} = \frac{n-1}{a_1a_n}$$

Sol: By considering a as the first term and d as the common difference, we can write a_n as a + (n – 1) d, where n = 1, 2, 3,... n.

$$\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \ldots.. + \frac{1}{a_{n-1}a_n} = \frac{1}{a(a+d)} + \frac{1}{(a+d)(a+2d)} + \ldots\ldots \frac{1}{[a+(n-2)d][a+(n-1)d]}$$

$$= \frac{1}{d}\left(\frac{1}{a} - \frac{1}{a+d}\right) + \frac{1}{d}\left(\frac{1}{a+d} - \frac{1}{a+2d}\right) + \ldots + \frac{1}{d}\left(\frac{1}{a+(n-2)d} - \frac{1}{a+(n-1)d}\right)$$

$$= \frac{1}{d}\left(\frac{1}{a} - \frac{1}{a+(n-1)d}\right) = \frac{a+(n-1)d-a}{ad(a+(n-1)d)} = \frac{(n-1)d}{ad(a+(n-1)d)} = \frac{n-1}{a(a+(n-1)d)} = \frac{n-1}{a_1a_n}$$

MASTERJEE CONCEPTS

Facts:

- If each term of an AP is increased, decreased, multiplied or divided by the same non-zero number, the resulting sequence is also an AP.
- The sum of the two terms of an AP equidistant from the beginning and end is constant and is equal to the sum of the first and last terms.

  $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots.$

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Illustration 14: Split 69 into three parts such that they are in AP and the product of the two smaller parts is 483. **(JEE MAIN)**

Sol: By considering the three parts as a – d, a, and a + d and using the given conditions, we can solve the given problem.

Sum of the three terms = 69 $\Rightarrow$ (a – d) + a + (a + d) = 69

$\Rightarrow$ 3a = 69 $\Rightarrow$ a = 23 ..... (i)

Product of the two smaller parts = 483 $\Rightarrow$ a (a – d) = 483

$\Rightarrow$ 23(23 – d) = 483 $\Rightarrow$ 23 – d = 21 $\Rightarrow$ d = 23 – 21 = 2

Hence, the three parts are 21, 23 and 25.

Illustration 15: Divide 32 into four parts that are in AP such that the ratio of the product of extremes to the product of mean is 7: 15. **(JEE MAIN)**

Sol: We can consider the four parts as (a – 3d), (a – d), (a + d) and (a + 3d).

Sum of the four parts = 32

$\Rightarrow$ (a – 3d) + (a – d) + (a + d) + (a + 3d) = 32 $\Rightarrow$ 4a = 32 $\Rightarrow$ a = 8

And $\frac{(a-3d)(a+3d)}{(a-d)(a+d)}=\frac{7}{15} \Rightarrow \frac{a^2-9d^2}{a^2-d^2}=\frac{7}{15} \Rightarrow \frac{64-9d^2}{64-d^2}=\frac{7}{15}$

$\Rightarrow 128\ d^2 = 512 \Rightarrow d^2 = 4 \Rightarrow d = \pm 2$

Thus, the required parts are 2, 6, 10 and 14.

Illustration 16: If a + b + c $\neq$ 0 and $\frac{b+c}{a},\frac{c+a}{b},\frac{a+b}{c}$ are in AP, then prove that $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are also in AP. **(JEE ADVANCED)**

Sol: Here $\frac{b+c}{a},\frac{c+a}{b},\frac{a+b}{c}$ are in AP; therefore, by adding 1 to each term and then by dividing each term by a + b +c, we will get the required result $\frac{b+c}{a},\frac{c+a}{b},\frac{a+b}{c}$ are in AP

Adding 1 to each term, find that $\left(\frac{b+c}{a}+1\right),\left(\frac{c+a}{b}+1\right),\left(\frac{a+b}{c}+1\right)$ are in AP

i.e. $\frac{a+b+c}{a},\frac{c+a+b}{b},\frac{a+b+c}{c}$ are in AP

Dividing each term by a + b + c, we find that $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in AP

Illustration 17: If a $\left(\frac{1}{b}+\frac{1}{c}\right),b\left(\frac{1}{a}+\frac{1}{c}\right),c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in AP, then prove that a, b, c are in AP. **(JEE ADVANCED)**

Sol: By adding 1 and then multiplying by $\left(\frac{abc}{ab+bc+ac}\right)$ to each term, we will get the result.

$$a\left(\frac{1}{b}+\frac{1}{c}\right),\ b\left(\frac{1}{a}+\frac{1}{c}\right),\ c\left(\frac{1}{a}+\frac{1}{b}\right) \text{ are in AP } \Rightarrow a\left(\frac{b+c}{bc}\right),\ b\left(\frac{a+c}{ac}\right),\ c\left(\frac{b+a}{ab}\right) \text{ are in AP}$$

Adding 1, we find that $\frac{ab+ac}{bc}+1,\frac{ab+bc}{ac}+1,\frac{bc+ac}{ab}+1$ are in AP

$\Rightarrow \frac{ab+ac+bc}{bc},\frac{ab+bc+ac}{ac},\frac{bc+ac+ab}{ab}$ are in AP

Multiplying by $\left(\frac{abc}{ab+bc+ac}\right)$ to all the terms, we find that a, b, c are in AP

4.3 Arithmetic Mean

The arithmetic mean (AM) A of any two numbers a and b is given by the equation (a + b)/2. Please note that the sequence a, A, b is in AP. If $a_1, a_2, \ldots, a_n$ are n numbers, the (AM) A, of these numbers is given by:

$$A = \frac{1}{n}(a_1 + a_2 + \ldots + a_n)$$

Inserting 'n' AMs between 'a' and 'b'

Suppose $A_1, A_2, A_3, \ldots, A_n$ be the n means between a and b. Thus, a, $A_1, A_2, \ldots, A_n$, b is an AP and b is the $(n + 2)^{th}$ term.

Thus, $b = a + (n + 1)d \quad \Rightarrow \quad d = \frac{b-a}{n+1}$

Now,

$$\begin{aligned} A_1 &= a + d \\ A_2 &= a + 2d \\ &\vdots \\ A_n &= a + nd \end{aligned}$$

$$\sum_{i=1}^{n} A_i = na + (1+2+3+\ldots+n)d = na + \left(\frac{n(n+1)}{2}\right)d = na + \left(\frac{n(n+1)}{2}\right)\left(\frac{b-a}{n+1}\right)$$

$$= \frac{n}{2}[2a + b - a] = nA \text{ where, } A = \frac{a+b}{2}$$

Note: The sum of the n AMs inserted between a and b is equal to n times A.M. between them.

Illustration 18: Insert 20 AMs between the numbers 4 and 67. **(JEE MAIN)**

Sol: Given, a = 4 and b = 67; therefore by using the formula $d = \frac{b-a}{n+1}$, we can solve it.

$$d = \frac{67-4}{20+1} = 3$$

$A_1 = a + d \quad \Rightarrow \quad A_1 = 7$

$A_2 = a + 2d \quad \Rightarrow \quad A_2 = 10$

$A_3 = a + 3d \quad \Rightarrow \quad A_3 = 13$

$A_{20} = a + 20d \quad \Rightarrow \quad A_{20} = 63$

Thus, between 4 and 67, 20 AMs are 7, 10, 13, 16, ..., 63.

Illustration 19: If $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the A.M. between a and b, find the value of n. **(JEE ADVANCED)**

Sol: Since the A.M. between a and b = $\frac{a+b}{2}$, we can obtain the value of n by equating this to $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$.

$\Rightarrow \quad \frac{a^n + b^n}{a^{n-1} + b^{n-1}} = \frac{a+b}{2}$ [Given] $\quad \Rightarrow \quad 2a^n + 2b^n = a^n + ab^{n-1} + a^{n-1}b + b^n$

$\Rightarrow \quad a^n - a^{n-1}b = ab^{n-1} - b^n \quad \Rightarrow \quad a^{n-1}(a - b) = b^{n-1}(a - b)$

$\Rightarrow \quad a^{n-1} = b^{n-1}$ [$\because a \neq b$] $\quad \Rightarrow \quad \left(\frac{a}{b}\right)^{n-1} = 1 = \left(\frac{a}{b}\right)^0 \quad \left(\because \left(\frac{a}{b}\right)^0 = 1\right)$

$\Rightarrow \quad n - 1 = 0 \quad \Rightarrow \quad n = 1$

Illustration 20: Between 1 and 31, m arithmetic means are inserted in such a way that the ratio of the 7^{th} and $(m-1)^{th}$ means is 5: 9. Calculate the value of m. **(JEE MAIN)**

Sol: AMs inserted between 1 and 31 are in AP. Thus, by considering d to be the common difference of AP and obtaining the 7^{th} and $(m-1)^{th}$ means we can solve the problem.

Suppose $A_1, A_2, A_3, A_4, \ldots\ldots A_m$ be the m AMs between 1 and 31.

Thus, 1, $A_1, A_2, \ldots. A_m$, 31 are in AP

The total number of terms is m + 2 and $T_{m+2} = 31$

$1 + (m + 2 - 1)\,d = 31 \quad \Rightarrow (m + 1)\,d = 30 \Rightarrow d = \dfrac{30}{m+1}$

$A_7 = T_8 = a + 7d = 1 + 7 \times \dfrac{30}{m+1} = \dfrac{m+1+210}{m+1} = \dfrac{m+211}{m+1}$

$A_{m-1} = T_m = 1 + (m-1)\,d = 1+(m-1)\times\dfrac{30}{m+1} = \dfrac{m+1+30m-30}{m+1} = \dfrac{31m-29}{m+1}$

$\dfrac{A_7}{A_{m-1}} = \dfrac{(m+211)/(m+1)}{(31m-29)/(m+1)} = \dfrac{m+211}{31m-29} = \dfrac{5}{9}$ [Given]

$\Rightarrow \dfrac{m+211}{31m-29} = \dfrac{5}{9} \Rightarrow 9m + 1899 = 155m - 145$

$\Rightarrow 146m = 2044 \Rightarrow m = \dfrac{2044}{146} = 14$;Thus, m = 14

Illustration 21: Gate receipts at the show of "Baghbaan" amounted to Rs 9500 on the first night and showed a drop of Rs 250 every succeeding night. If the operational expenses of the show are Rs 2000 a day, find out on which night the show ceases to be profitable? **(JEE MAIN)**

Sol: Here, a = 9500 and d = –250. The show ceases to be profitable on the night when the receipts are just Rs 2000. Thus, by considering that it will happen at n^{th} night and using $T_n = a + (n-1)\,d$, we can solve this problem.

We have the cost of gate receipt on the first night (a) = 9500

Common difference (d) = –250

Suppose, it happens on the nth night, then

$2000 = 9500 + (n-1)(-250) \Rightarrow 2000 - 9500 = -250n + 250$

$\Rightarrow -7500 - 250 = -250n \Rightarrow -7750 = -250n \Rightarrow n = \dfrac{7750}{250} = 31$

MASTERJEE CONCEPTS

(a) If the sum of n terms S_n is given, then the general term $T_n = S_n - S_{n-1}$, where S_{n-1} is sum of (n – 1) terms of AP.

(b) In a series, if S_n is a quadratic function of n or T_n is a linear function of n, the series is an AP.

(i) If $T_n = an + b$, the series so formed is an AP and its common difference is a.

(ii) If $S_n = an^2 + bn + c$, the series so formed is an AP and its common difference is 2a.

MASTERJEE CONCEPTS

(c) If in a finite AP, the number of terms is odd, then its middle term is the A.M. between the first term and last term and its sum is equal to the product of the middle term and number of terms.

(d) It is found that the sum of infinite terms of an AP is ∞, if d > 0 and –∞, if d < 0.

(e) If for an AP, the pth term is q and the qth term is p, then the mth term is = p + q – m.

(f) If for an AP, the sum of p terms is q and sum of q terms is p, then the sum of (p + q) terms is – (p + q).

(g) If for an AP, the sum of p terms is equal to the sum of q terms, then the sum of (p +q) terms is zero.

(h) If for different APs, $\frac{S_n}{S'_n}=\frac{f_n}{\phi_n}$, then $\frac{T_n}{T'_n}=\frac{f(2n-1)}{\phi(2n-1)}$.

(i) If for two APs, $\frac{T_n}{T'_n}=\frac{An+B}{Cn+D}$, then we find that $\frac{S_n}{S'_n}=\frac{A\left(\frac{n+1}{2}\right)+B}{C\left(\frac{n+1}{2}\right)+D}$.

Shrikant Nagori (JEE 2009, AIR 54)

An Important Property of AP: A sequence is said to be an AP if the sum of its n terms is of the form $An^2 + Bn$, where A and B are constants. Thus, the common difference of the AP is 2A.

Proof: Suppose, a and d be the first term and common difference of AP, respectively, and S_n be the sum of n terms.

$$S_n=\frac{n}{2}[2a+(n-1)d]$$

$$\Rightarrow \quad S_n = an+\frac{n^2}{2}d-\frac{n}{2}d=\left(\frac{d}{2}\right)n^2+\left(a-\frac{d}{2}\right)n$$

$$\Rightarrow \quad S_n = An^2 + Bn, \text{ where } A = \frac{d}{2} \text{ and } B = a-\frac{d}{2}$$

Hence, the sum of n terms of an AP is of the form $An^2 + Bn$.

Conversely, suppose the sum S_n of n terms of a sequence $a_1, a_2, a_3,$ ……….a_n….. is of the form $An^2 + Bn$. Then, we have to prove that the sequence is an AP.

We have $S_n = An^2 – Bn$

$\Rightarrow$ $S_{n-1} = A(n – 1)^2 + B(n – 1)$ [On replacing n by n = 1]

Now, $A_n = S_n – S_{n–1}$

$\Rightarrow$ $A_n = \{An^2 + Bn\} – \{A(n–1)^2 – B(n–1)\} = 2An + (B – A)$

$\Rightarrow$ $A_{n+1} = 2A (n+1) + (B – A)$ [On replacing n by (n + 1)]

$\therefore$ $A_{n+1} – A_n = \{2A(n + 1) +B – A\} – \{2An + (B – A)\} = 2A$

Since $A_{n+1} – A_n = 2A$ for all $n \in N$, the sequence is an AP with a common difference 2A.

For example, if $S_n = 3n^2 + 2n$, we can say that it is the sum of the n terms of an AP with a common difference of 6.

5. GEOMETRIC PROGRESSION

A sequence of non-zero numbers is called a geometric progression (GP) if the ratio of successive terms is constant.

In general, G.P. is written in the following form: a, ar, ar^2, ..., ar^{n-1},....

where a is the first term and r is the common ratio

5.1 General Term

If a is the first term and r is the common ratio, then $T_n = ar^{n-1}$.

Illustration 22: The 5th, 8th and 11th terms of a G.P. are given as p, q and, s respectively. Prove that $q^2 = ps$.

(JEE MAIN)

Sol: By using $T_n = ar^{n-1}$ and solving it, we can prove the problem.

Given, $T_5 = p, T_8 = q, T_{11} = s$ ....(i)

Now, $T_5 = ar^{5-1} = ar^4 \Rightarrow ar^4 = p$ .... (ii) [Using (i)]

$T_8 = ar^{8-1} = ar^7 \Rightarrow ar^7 = q$ ... (iii) [Using (i)]

$T_{11} = ar^{11-1} = ar^{10} \Rightarrow ar^{10} = s$ ..... (iv)

On squaring (iii), we get

$q^2 = a^2 r^{14} = a\cdot a.r^4\cdot r^{10} = (ar^4)(ar^{10})$

$\Rightarrow$ $q^2 = ps$ [Using (ii) and (iv)] proved.

5.2 Series of GP

Let us suppose $S_n = a + ar + ar^2 + \ldots\ldots + ar^{n-1}$ ... (i)

Multiplying 'r' on both the sides of (i) and shifting the RHS terms by one place, we get

$S_n r = 0 + ar + ar^2 + \ldots\ldots + ar^n$ ... (ii)

By subtracting (ii) from (i), we get

$S_n(1 - r) = a - ar^n = a(1 - r^n)$

$S_n = \frac{a(1-r^n)}{1-r}$, where $r \neq 1$

Thus, the sum of the first n terms of a G.P. is given by $\Rightarrow S_n = \left(\frac{a(r^n-1)}{r-1}\right) = \left(\frac{T_{n+1}-a}{r-1}\right)$

And $S_n = na$, when r = 1

Note: If r = 1, then the sequence is of both AP and GP, and its sum is equal to na, i.e. $S_n = na$.

If $|r| < 1$, the nth term of G.P. converges to zero and the sum becomes finite.

The sum to infinite terms of G.P. = $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(r^n-1)}{r-1}$

As $|r| < 1$ $r^n \to 0$ as $n \to \infty$ $\therefore$ $S_\infty = \frac{a}{1-r}$

5.3 Geometric Mean

If a, b and c are three positive numbers in GP, then b is called the geometrical mean (GM) between a and c, and $b^2 = ac$. If a and b are two real numbers of the same sign and G is the G.M. between them, $G^2 = ab$.

Note: If a and b are two number of opposite signs, then the G.M. between them does not exist.

To Insert 'n' GMs Between a and b: If a and b are two positive numbers and we have to insert n GMs, G_1, G_2, ..., G_n, between the two numbers 'a' and 'b' then a, G_1, G_2, ..., G_n, b will be in GP. The series consists of (n + 2) terms and the last term is b and the first term is a.

Thus, $b = ar^{n+2-1} \Rightarrow b = ar^{n+1} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$\Rightarrow \quad G_1 = ar,\ G_2 = ar^2 \ ....G_n = ar^n$ or $G_n = a^{yn+1} \cdot b^{yn+1} = (ab)^{yn+1}$

Note: The product of n GMs inserted between 'a' and 'b' is equal to the n^{th} power of the single G.M. between 'a' and 'b,' i.e.

$\prod_{r=1}^{n} G_r = (G)^n$, where $G = \sqrt{ab}$ (GM between a and b)

5.4 Relation between A.M. and GM

For any two non-negative number A.M. ≥ G.M.

Proof: Let two non-negative numbers be $\sqrt{a}$ and $\sqrt{b}$.

Now, we can write $\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0 \Rightarrow a - 2\sqrt{ab} + b \geq 0 \quad \Rightarrow a + b \geq 2\sqrt{ab} \Rightarrow \frac{a+b}{2} \geq \sqrt{ab} \quad \Rightarrow$ A.M. ≥ GM

Note: (i) Equality for AM, G.M. (i.e. A.M. = GM) exists when a = b.

(ii) Since A.M. ≥ GM; $(AM)_{min}$ = GM; $(GM)_{max}$ = AM

Illustration 23: If x, y and z have the same sign, then prove that $\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geq 3$. **(JEE ADVANCED)**

Sol: As we know that A.M. ≥ G.M., therefore by obtaining A.M. and G.M. of $\frac{x}{y}$, $\frac{y}{z}$ and $\frac{z}{x}$ we can prove the problem.

Let $\frac{x}{y} = x_1$; $\frac{y}{z} = x_2$; $\frac{z}{x} = x_3$

$\therefore \ \frac{x_1+x_2+x_3}{3} \geq (x_1x_2x_3)^{1/3} \Rightarrow \frac{\frac{x}{y}+\frac{y}{z}+\frac{z}{x}}{3} \geq 1$

Hence proved.

Illustration 24: Calculate the values of n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the G.M. between a and b. **(JEE ADVANCED)**

Sol: We know that the G.M. between a and b = $\sqrt{ab}$, but here G.M. between a and b is $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$.

$\Rightarrow \quad \frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \sqrt{ab} \qquad \Rightarrow \quad a^{n+1} + b^{n+1} = (a^n + b^n)(ab)^{1/2}$

$\Rightarrow \quad a^{n+1} + b^{n+1} = a^{n+\frac{1}{2}} \cdot b^{\frac{1}{2}} + a^{\frac{1}{2}} \cdot b^{n+\frac{1}{2}} \qquad \Rightarrow a^{n+1} - a^{n+\frac{1}{2}} \cdot b^{\frac{1}{2}} = a^{\frac{1}{2}} \cdot b^{n+\frac{1}{2}} - b^{n+1}$

$\Rightarrow \quad a^{n+\frac{1}{2}}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) = b^{n+\frac{1}{2}}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) \qquad \Rightarrow a^{n+\frac{1}{2}} = b^{n+\frac{1}{2}}$

$\Rightarrow \quad \left(\frac{a}{b}\right)^{n+\frac{1}{2}} = 1 = \left(\frac{a}{b}\right)^0 \qquad \Rightarrow \quad n+\frac{1}{2} = 0 \Rightarrow \quad n = -\frac{1}{2}$

Illustration 25: Find the sum to n terms for the series 9 + 99 + 999 ...... n. **(JEE ADVANCED)**

Sol: The given series can be written as $S = (10 - 1) + (10^2 - 1) + (10^3 - 1) \ldots\ldots + (10^n - 1)$.

Thus, by using $S_n = \frac{a(1-r^n)}{1-r}$, we can find out the required sum.

$\therefore \quad S = (10 + 10^2 + 10^3 + \ldots + 10^n) - n; \quad S = \frac{10(1-10^n)}{1-10} - n$

Illustration 26: If a_1, a_2 and a_3 are in G.P. with a common ratio r (r > 0 and a > 0), then values of r for which inequality $9a_1 + 5a_3 > 14a_2$ hold good are? **(JEE ADVANCED)**

Sol: Since $a_1 = \frac{a}{r}$ $a_2 = a$, $a_3 = ar$, by substituting these values to the given inequality we will get the result.

$a_1 = \frac{a}{r}$, $a_2 = a$, $a_3 = ar$ Now, $\frac{9a}{r} + 5ar > 14a$

$\Rightarrow \quad 5ar^2 - 14ar + 9a > 0 \qquad \Rightarrow \quad 5r^2 - 14r + 9 > 0$

$\Rightarrow \quad 5r^2 - 5r - 9r + 9 > 0 \qquad \Rightarrow \quad 5r(r-1) - 9(r-1) > 0$

$\Rightarrow \quad (5r - 9)(r - 1) > 0 \qquad \Rightarrow \quad r \in R - \left(1, \frac{5}{9}\right)$

MASTERJEE CONCEPTS

- The product of n geometric means between a and (1/a) is 1.
- Let the first term of a G.P. be negative; if r > 1, then it is a decreasing G.P. and if 0< r < 1, then it is an increasing GP.
- If $a_1, a_2, a_3, \ldots, a_n$ are in AP, $a^{a_1}, a^{a_2}, a^{a_3}, \ldots, a^{a_n}$ will be in G.P. whose common ratio is a^d.

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Illustration 27: On a certain date, the height of a plant is 1.6 m. If the height increases by 5 cm in the following year and if the increase in each year is half of that in the preceding years, show that the height of the plant will never be 1.7 m. **(JEE MAIN)**

Sol: Here, the sum of the increases in the height of the plant in the first, second, third, ... year is equal to (1.7 – 1.6) m = 0.1 m = 10 cm.

According to the question, increases in the height of the plant in the first, second, third, ... year are 5, $\frac{5}{2}, \frac{5}{4}$, ... cm, respectively.

Let it reach the height of 1.7 m (i.e. increases [1.7 – 1.6] m = 0.1 m = 10 cm).

Therefore, the sum of 5, $\frac{5}{2}, \frac{5}{4}$, ...to n terms = 10

$$\Rightarrow \quad \frac{5\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} = 10 \qquad \left[\because a = 5,\ r = ½,\ S_n = \frac{a(1-r^n)}{1-r}\right]$$

$\Rightarrow \quad 10\left(1 - \frac{1}{2^n}\right) = 10 \Rightarrow 1 - \frac{1}{2^n} = 1 \Rightarrow \frac{1}{2^n} = 0$, which does not hold for any n. Thus, the plant will never reach the height of 1.7 m.

Illustration 28: A manufacturer reckons that the value of a machine (price = Rs 15,625) will depreciate each year by 20%. Calculate the estimated value at the end of 5 years. **(JEE MAIN)**

Sol: Here the value of the machine after 5 years= ar^5, where a= 15,625. We will obtain the value of r using the given condition.

The present value of the machine = Rs 15,625

The value of the machine in the next year = Rs 15,625 × $\frac{80}{100}$

The value of the machine after 2 years = Rs 15,625 × $\frac{80}{100} \times \frac{80}{100}$

The values of the machine in the present year, after 1 year and after 2 years are

Rs 15,625, Rs 15,625 × $\frac{80}{100}$ and Rs 15,625 × $\frac{80}{100} \times \frac{80}{100}$, respectively

These values form a GP.

Here, the first term is Rs 15,625 and the common ratio is $\frac{80}{100}$,i.e. $\frac{4}{5}$.

Thus, thee value of the machine after 5 years = ar^5 = Rs 15,625 $\times \left(\frac{4}{5}\right)^5 = \frac{15625 \times 1024}{625 \times 5} = 1024 \times 5$ = Rs 5120

5.5 Properties of GP

(a) If each term of a G.P. is multiplied or divided by the same non-zero quantity, then the resulting sequence is also a GP.

(b) If in a finite GP, the number of terms is odd, then its middle term is the G.M. of the first and last terms.

(c) If a, b and c are in GP, then $\frac{b}{a} = \frac{c}{b} \Rightarrow b^2 = ac$ (which is the condition of GP).

(d) The reciprocals of the terms of a given G.P. also give a G.P. with a common ratio of $\frac{1}{r}$.

Proof: Let $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ be the terms of a G.P. with the common ratio r.

Then, $\frac{a_{n+1}}{a_n} = r$ for all $n \in N$ ... (i)

The sequence formed by the reciprocals of the terms of the above G.P. is given by

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}, \ldots$$

Now, $\frac{1/a_{n+1}}{1/a_n} = \frac{a_n}{a_{n+1}} = \frac{1}{r}$ [Using (i)]

Hence, the new sequence is also a G.P. with the common ratio 1/r.

(e) If each term of a G.P. is raised to the same power (say k), then the resulting sequence also forms a G.P. with the common ratio as r^k.

Proof: Let $a_1, a_2, a_3, a_4, \ldots, a_n \ldots$ be the terms of a G.P. with the common ratio r.

Then, $\frac{a_{n+1}}{a_n} = r$ for all $n \in N$ (i)

Let k be a non-zero real number. Consider the sequence. $a_1^k, a_2^k, a_3^k, \ldots, a_n^k, \ldots$

Here, $\frac{a_{n+1}^k}{a_n^k} = \left(\frac{a_{n+1}}{a_n}\right)^k = r^k$ for all $n \in d$ [Using (i)]

Thus, $a_1^k, a_2^k, a_3^k, ..., a_n^k, ...$ is a G.P. with a common ratio r^k.

(f) In a GP, the product of the terms equidistant from the beginning and the end is always the same and it is equal to the product of the first and last terms (only for finite GP).

Proof: Let a_1, a_2, a_3 ,......, a_n be a finite G.P. with the common ratio r. Then,

k^{th} term from the beginning = $a_k = a_1 r^{k-1}$

k^{th} term from the end = $(n - k + 1)^{th}$ term from the beginning

$= a_{n-k+1} = a_1 r^{n-k}$

$\therefore$ (k^{th} term from the beginning) (k^{th} term from the end)= $a_k a_{n-k+1}$

$= a_1 r^{k-1} a_1 r^{n-k} = a_1^2 r^{n-1} = a_1 . a_1 r^{n-1} = a_1 a_n$ for all k = 2,3,....,n –1

Thus, the product of the terms equidistant from the beginning and the end is always the same and is equal to the product of the first and last terms.

(g) If the terms of a G.P. are chosen at regular intervals, the new sequence so formed also forms a G.P. with the common ratio as r^p , where p is the size of interval.

For example:

2, 4, 8, 16, 32, 64, 128,..... (GP, where r = 2)

4, 16, 64 ..... (also a GP, where r = 4)

(h) If a_1, a_2, a_3 ,...., a_r.... is a G.P. of non-zero, non-negative terms, then log a_1, log a_2,...., log a_n,..... is an AP and vice versa.

(i) If T_1, T_2, T_3... and t_1, t_2, t_3 are two GPs, T_1t_1, T_2t_2, T_3t_3.... is also in GP.

Proof: Let the two GPs be T_1, T_2, ..., T_n,... with the common ratio R

$$\Rightarrow \frac{T_{n+1}}{T_n} = R \qquad ... (i)$$

and t_1, t_2,....,t_n,.... with common the ratio r

$$\Rightarrow \frac{t_{n+1}}{t_n} = r \qquad .... (ii)$$

Multiplying each term of the sequence (i) by the corresponding term of (ii), we get $\left(\frac{T_{n+1}}{T_n}\right)\left(\frac{t_{n-1}}{t_n}\right) = Rr$

Thus, the resulting sequence is also in G.P. with the common ratio Rr.

(j) The resulting sequence thus formed by dividing the terms of a G.P. by the corresponding terms of another G.P. is also a GP.

Proof: Let the two GPs be T_1,T_2,....,T_n,.... with the common ratio R

$$\Rightarrow \frac{T_{n+1}}{T_n} = R \qquad ... (i)$$

and t_1,t_2,....,t_n.... with the common ratio r

$$\Rightarrow \frac{t_{n+1}}{t_n} = r \qquad ...(ii)$$

Dividing each term of the sequence (i) by the corresponding term of (ii), we get

$$\frac{\frac{T_{n+1}}{t_{n+1}}}{\frac{T_n}{t_n}} = \left(\frac{R}{r}\right) \qquad \text{[Using (i) and (ii)]}$$

Thus, the resulting sequence is also in G.P. with the common ratio $\left(\frac{R}{r}\right)$.

Illustration 29: If sum of infinite terms of G.P. is 15 and sum of squares of infinite terms of G.P. is 45, then find GP.

(JEE MAIN)

Sol: As the sum of infinite terms $S_\infty = \frac{a}{1-r}$, therefore by using this formula we can obtain the value of the common ratio.

$$\frac{a}{1-r} = 15$$

Now, $a^2, a^2r^2, a^2r^4, \ldots\ldots;$ $\quad \frac{a^2}{1-r^2} = 45 \quad \therefore \quad \frac{225(1-r)(1-r)}{(1-r)(1+r)} = 45$

$225 - 225r = 45 + 45r; \qquad 180 = 270\,r \quad \therefore \quad r = 2/3$

Illustration 30: If $x = 1 + a + a^2 + \ldots. \infty$, $y = 1 + b + b^2 + \ldots. \infty$ and $|a| < 1$, $|b| < 1$, then prove that $1 + ab + a^2b^2 + \ldots. = \frac{xy}{x+y-1}$

(JEE MAIN)

Sol: By using the formula $S_\infty = \frac{a}{1-r}$, we can solve problem.

$x = 1 + a + a^2 + \ldots. \text{ to } \infty = \frac{1}{1-a} \qquad (\because |a| < 1)$

$$\Rightarrow \quad 1 - a = \frac{1}{x} \Rightarrow a = 1 - \frac{1}{x} \quad \Rightarrow a = \frac{x-1}{x} \qquad \ldots \text{(i)}$$

Also, $y = 1 + b + b^2 + \ldots\ldots \text{ to } \infty = \frac{1}{1-b} \qquad (\because |b| < 1)$

$$\Rightarrow \quad 1 - b = \frac{1}{y} \Rightarrow b = 1 - \frac{1}{y} \quad \Rightarrow b = \frac{y-1}{y} \qquad \ldots \text{(ii)}$$

$\therefore \quad 1 + ab + a^2b^2 + \ldots.. \text{ to } \infty = \frac{1}{1-ab} \qquad (\because |a| < 1, |b| < 1 \Rightarrow |ab| < 1)$

$$= \frac{1}{1 - \frac{x-1}{x}\cdot\frac{y-1}{y}} \text{ [Using (i) and (ii)]} = \frac{xy}{xy - xy + x + y - 1} = \frac{xy}{x+y-1}$$

Hence proved.

Illustration 31: If $S_1, S_2, S_3, \ldots\ldots, S_p$ denote the sum of an infinite G.P. whose first terms are 1, 2, 3, ... , p, respectively and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots, \frac{1}{(p+1)}$, respectively, show that $S_1 + S_2 + S_3 + \ldots\ldots + S_p = \frac{p(p+3)}{2}$.

(JEE ADVANCED)

Sol: By using $S_\infty = \frac{a}{1-r}$ we can obtain $S_1, S_2, S_3, \ldots\ldots, S_p$ and after that by adding them we can prove the given equation.

For S_1, we have $a = 1$, $r = \frac{1}{2}$ $\therefore$ $S_1 = \frac{1}{1-\frac{1}{2}} = 2$

For S_2, we have $a = 2$, $r = \frac{1}{3}$ $\therefore$ $S_2 = \frac{2}{1-\frac{1}{3}} = 3$

For S_3, we have $a = 3$, $r = \frac{1}{4}$ $\therefore$ $S_3 = \frac{3}{1-\frac{1}{4}} = 4$

For S_p, we have $a = p$, $r = \frac{1}{p+1}$ $\therefore$ $S_p = \frac{p}{1-\frac{1}{p+1}} = p + 1$

Adding all these, we get $S_1 + S_2 + S_3 + \ldots\ldots\ldots + S_p = 2 + 3 + 4 + \ldots\ldots\ldots + (p + 1)$

$$= \frac{p}{2}[2 + (p + 1)] = \frac{p}{2}[p + 3] = \frac{p(p+3)}{2}$$

Hence proved.

6. ARITHMETIC GEOMETRIC PROGRESSION

A series formed by multiplying the corresponding terms of AP and G.P. is called arithmetic geometric progression (AGP).

Let a = first term of AP, b = first term of GP, d = common difference and r = common ratio of GP, then

AP: $a, a + d, a + 2d, a + 3d, \ldots\ldots, a + (n - 1) d$

GP: $b, br, br^2, br^3, \ldots\ldots, br^{n-1}$

AGP: $ab, (a + d) br, (a + 2d) br^2 \ldots. (a + (n - 1) d) br^{n-1}$ (Standard appearance of AGP)

The general term (n^{th} term) of an AGP is given as $T_n = [a + (n - 1)d] br^{n-1}$.

6.1 Series of AGP

To find the sum of n terms of an AGP, we suppose that its sum as S_n and then multiply both the sides by the common ratio of the corresponding G.P. and then subtract as in the following way. Thus, we get a G.P. whose sum can be easily obtained.

$S_n = ab + (a + d) br + (a + 2d) br^2 + \ldots\ldots + (a + (n-1)d) br^{n-1}$ ...(i)

$rS_n = 0 + abr + (a + d) br^2 + \ldots\ldots + (a + (n - 1)d) br^n$ ...(ii)

After subtraction, we get

$S_n(1 - r) = ab + [dbr + dbr^2 + \ldots\ldots + \text{ up to } (n - 1) \text{ terms}] - [(a + (n - 1)d)br^n]$

$$S_n(1 - r) = ab + \frac{dbr(1-r^{n-1})}{1-r} - (a+(n-1)d)br^n$$

$$S_n = \frac{ab}{1-r} + \frac{dbr(1-r^{n-1})}{(1-r)^2} - \frac{(a+(n-1)d)br^n}{1-r}$$. This is the sum of n terms of AGP

For an infinite AGP, as $n \to \infty$, then $r^n \to 0$ ($\because |r| < 1$)

$$\Rightarrow \quad S_\infty = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$$

Illustration 32: If $|x| < 1$, then find the sum $S = 1 + 2x + 3x^2 + 4x^3 \ldots\ldots + \infty$. **(JEE MAIN)**

Sol: The sum can be found out by calculating the value of $Sx - S$.

$Sx = x + 2x^2 + 3x^3 + 4x^3 + \ldots\ldots\infty$

$S(1 - x) = 1 + x + x^2 + x^3 + \ldots\ldots + \infty$; $\quad S(1 - x) = \dfrac{1}{(1-x)} \Rightarrow S = \dfrac{1}{(1-x)^2}$

Illustration 33: If $|x| < 1$, then find the sum $S = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots\infty$. **(JEE ADVANCED)**

Sol: Similar to above illustration.

$S = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots\infty$

$Sx = x + 3x^2 + 6x^3 \ldots\ldots\ldots\ldots\infty$

$S(1-x) = 1 + 2x + 3x^2 + 4x^3 \ldots\ldots\ldots\ldots \infty$

$S(x)(1-x) = x + 2x^2 + 3x^3 \ldots\ldots\ldots \infty$

$S(1-x)^2 = 1 + x + x^2 + \ldots\ldots \infty$

$S(1 - x)^2 = \dfrac{1}{1-x} \quad S = \dfrac{1}{(1-x)^3}$

7. MISCELLANEOUS SEQUENCES

Type 1: Some Standard Results

(i) Sum of the first n natural numbers $= \displaystyle\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

(ii) Sum of the first n odd natural numbers $= \displaystyle\sum_{r=1}^{n} (2r-1) = n^2$

(iii) Sum of the first n even natural numbers $= \displaystyle\sum_{r=1}^{n} 2r = n(n+1)$

(iv) Sum of the squares of the first n natural numbers $= \displaystyle\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

Proof: $\displaystyle\sum_{n=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}$

Consider $(x + 1)^3 = x^3 + 1 + 3x^2 + 3x$

$(x + 1)^3 - x^3 = 3x^2 + 3x + 1$

Put $x = 1, 2, 3 \ldots. n$

$2^3 - 1^3 = 3.1^2 + 3.1 + 1$

$3^3 - 2^3 = 3.2^2 + 3.2 + 1$

$(n + 1)^3 - n^3 = 3n^2 + 3.n + 1$

Adding all, we get

$\Rightarrow \quad (n + 1)^3 - 1 = 3(1^2 + 2^2 + 3^2 + \ldots. + n^2) + 3(1 + 2 + \ldots. + n) + n$

$\Rightarrow \quad (n + 1)^3 - 1 = 3\Sigma n^2 + 3\dfrac{3n(n+1)}{2} + n \Rightarrow \quad 3\Sigma n^2 = (n + 1)^3 - 1 - \dfrac{3n(n+1)}{2} - n$

$\Rightarrow \quad 3\Sigma n^2 = \quad n^3 + 1 + 3n^2 + 3n - 1 - \dfrac{3n(n+1)}{2} - n$

$\Rightarrow \quad 3\Sigma n^2 = n^3 + 3n^2 + 2n - \dfrac{3n(n+1)}{2}$

$\Rightarrow \quad 3\Sigma n^2 = \dfrac{2n^3 + 6n^2 + 4n - 3n^2 - 3n}{2} \Rightarrow 3\Sigma n^2 = \dfrac{2n^3 + 3n^2 + n}{2}$

$\Rightarrow \quad 3\Sigma n^2 = \dfrac{2n^3 + 3n^2 + n^2 + n}{2} \Rightarrow 3\Sigma n^2 = \dfrac{2n^2(n+1) + n(n+1)}{2}$

$\Rightarrow \quad 3\Sigma n^2 = \dfrac{n(n+1)\times(2n+1)}{2} \Rightarrow \Sigma n^2 = \dfrac{n(n+1)\times(2n+1)}{6}$

(v) Sum of the cubes of first n natural numbers $\sum_{r=1}^{n} r^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

Proof: Consider $(x + 1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1$

Put $x = 1, 2, 3 \ldots\ldots n$

$2^4 - 1^4 = 4\cdot1^3 + 6\cdot1^2 + 4\cdot1 + 1$

$3^4 - 2^4 = 4\cdot2^3 + 6\cdot2^2 + 4\cdot2 + 1$

$4^4 - 3^4 = 4\cdot3^2 + 6\cdot2^2 + 4\cdot3 + 1$

:

$(n + 1)^4 - n^4 = 4\cdot n^3 + 6\cdot n^2 + 4\cdot n + 1$

Adding all, we get

$(n + 1)^4 - 1^4 = 4(1^3 + 2^3 + \ldots\ldots + n^3) + 6(1^2 + 2^2 + \ldots.+ n^2) + 4(1 + 2 + 3\ldots\ldots +n) +n$

$= 4\Sigma n^3 + 6\left(\dfrac{n(n-1)(2n+1)}{6}\right) + 4\left(n\left(\dfrac{n+1}{2}\right)\right) + n$

On simplification, we get

$\Sigma n^3 = \left(n\left(\dfrac{n+1}{2}\right)\right)^2$

(vi) Sum of the fourth powers of the first n natural numbers (Σn^4)

$\Sigma n^4 = 1^4 + 2^4 + \ldots\ldots +n^4 \quad ; \quad \Sigma n^4 = \dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

[The result can be proved in the same manner as done for Σn^3]

Illustration 34: Find the value of $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}(1)$. **(JEE MAIN)**

Sol: Using the formula $\sum_{r=1}^{n} r = \dfrac{n(n+1)}{2}$ and $\sum_{r=1}^{n} r^2 = \dfrac{n(n+1)(2n+1)}{6}$, we can solve the problem.

Let $S = \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}(1)$

$S = \sum_{i=1}^{n}\sum_{j=1}^{i}(j) = \sum_{i=1}^{n}\dfrac{i(i+1)}{2} = \dfrac{1}{2}[\Sigma n^2 + \Sigma n] = \dfrac{1}{2}\left[\dfrac{n(n+1)(2n+1)}{6} + \dfrac{n(n+1)}{2}\right] = \dfrac{n(n+1)(2n+4)}{12} = \dfrac{n(n+1)(n+2)}{6}$

Illustration 35: Find the sum of 1.2.3 + 2.3.4 + 3.4.5...... n terms. **(JEE MAIN)**

Sol: The given series is in the form of $T_n = n(n + 1)(n + 2) = n^3 + 3n^2 + 2n$.

Therefore, by using $\sum_{r=1}^{n} r^3 = n^2\left(\frac{n+1}{2}\right)^2$ $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$, we can solve the problem.

$T_n = n(n + 1)(n + 2) = n(n^2 + 3n + 2)$

$T_n = n^3 + 3n^2 + 2n$

$\Sigma T_n = \Sigma n^3 + 3\Sigma n^2 + 2\Sigma n = n^2\left(\frac{n+1}{2}\right)^2 + \frac{3n(n+1)(2n+1)}{6} + n(n+1)$

Type 2: Using Method of Difference: If $T_1, T_2, T_3, T_2, T_4 T_5$.... is a sequence whose terms are sometimes in AP and sometimes in GP, then for such series we first compute their nth term and then compute the sum to n terms using sigma notation.

Illustration 36: Find $S_n = 6 + 13 + 22 \text{.........} + T_n$. **(JEE ADVANCED)**

Sol: By calculating $[S_n + (-S_n)]$, we will get T_n. After that we will obtain ΣT_n and thus we will get the result.

$\Sigma_n = 6 + 13 + 22 \text{.........} T_n$

$-\Sigma_n = -6 - 13 \text{.........} - T_{n-1} - T_n$

$\Rightarrow \quad 0 = 6 + (7 + 9 + 11 \text{.....} (T_n - T_{n-1})) - T_n$

$\Rightarrow \quad T_n = 6 + (7 + 9 + 11 \text{....} (T_n - T_{n-1})) = 6 + (n - 1)(7 + n - 2) = 6 + (n - 1)(n + 5)$

$\Rightarrow \quad T_n = 6 + n^2 + 4n - 5 = n^2 + 4n + 1$

$\Sigma_n = \Sigma n^2 + 4\Sigma n + n = \frac{n(n+1)(n+1)}{6} + 2n(n+1) + n$

Illustration 37: Find $S = 1+\left(1+\frac{1}{3}\right)+\left(1+\frac{1}{3}+\frac{1}{3^2}\right)+\text{......}$ n terms. **(JEE ADVANCED)**

Sol: Given, $T_n = 1+\frac{1}{3}+\frac{1}{3^2}+\text{.....}+\frac{1}{3^{n-1}}$; therefore by obtaining ΣT_n, we will get the result.

$S = 1+\left(1+\frac{1}{3}\right)+\left(1+\frac{1}{3}+\frac{1}{9}\right)\text{.......}$

$T_n = 1+\frac{1}{3}+\frac{1}{3^2}+\text{.....}+\frac{1}{3^{n-1}} = \frac{3\left(1-\frac{1}{3^n}\right)}{2}$

$\Sigma T_n = \frac{3n}{2}-\frac{3}{2}\Sigma\frac{1}{3^n} = \frac{3n}{2}-\frac{3}{2}\left(\frac{1}{3}+\frac{1}{3^2}\text{......}\frac{1}{3^n}\right) = \frac{3}{2}\left(n-\frac{3}{2}\left(1-\frac{1}{3^n}\right)\right)$

Type 3: Splitting the n^{th} term as a difference of two: Here, S is a series in which each term is composed of the reciprocal of the product of r factors in an AP.

Illustration 38: Find the sum of n terms of the series $\frac{1}{1\cdot2\cdot3\cdot4}+\frac{1}{2\cdot3\cdot4\cdot5}+\frac{1}{3\cdot4\cdot5\cdot6}+.....$ **(JEE ADVANCED)**

Sol: Here nth term of the series will be $T_n = \frac{1}{n(n+1)(n+2)(n+3)}$.

By considering $S_n = c - \lambda$, where $\lambda = \frac{1}{3(n+1)(n+2)(n+3)}$, we will get the result.

First calculate the nth term, $T_n = \frac{1}{n(n+1)(n+2)(n+3)}$

Now, let the sum of the above series be given by:

$S_n = c - \lambda$, where λ is obtained by replacing the first factor by (last factor – first factor) ... (i)

Hence,

$$\lambda = \frac{1}{3(n+1)(n+2)(n+3)} \quad \begin{bmatrix}\text{First factor} = n \\ \text{Last factor} = n+3\end{bmatrix} \qquad \text{... (ii)}$$

Using (ii) $\Rightarrow$ $S_n = c - \frac{1}{3(n+1)(n+2)(n+3)}$ ... (iii)

To calculate 'c', put n = 1 in (iii)

$$S_1 = c - \frac{1}{1\cdot2\cdot3\cdot4} \Rightarrow \frac{1}{1\cdot2\cdot3\cdot4} = c - \frac{1}{3\cdot2\cdot3\cdot4} \quad \Rightarrow \quad c = 1/18$$

Put the value of 'c' in (iii)

$$S_n = \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)},\ S_n = \frac{1}{3}\left\{\frac{1}{6} - \frac{1}{(n+1)(n+2)(n+3)}\right\}$$

Remark $\Rightarrow$ If we want to calculate S_∞, then $n \to \infty$, $\frac{1}{(n+1)(n+2)(n+3)} \to 0 \Rightarrow S_\infty = \frac{1}{18}$

Note: The above method is applicable only when the series looks like as follows:

$$\frac{1}{a(a+d)(a+2d)} + \frac{1}{(a+d)(a+2d)(a+3d)} + \frac{1}{(a+2d)(a+3d)(a+4d)} + ...$$

Type 4: Vn Method: This is method of resolving the nth term into partial fraction and summation by telescopic cancellation. First, find the nth term of the series and try to create a denominator part in the numerator by using partial fraction whenever the series is in the form of fraction or T_n is in the form of fraction.

For example, let us suppose a summation where the nth term is like the following:

$$T_n = \frac{2}{n^2-1}$$

Using the partial fraction, we can write the nth term as $T_n = \frac{1}{n-1} - \frac{1}{n+1}$

Now, when we find the summation, there will be telescopic cancellation and thus we will get the sum of the given series.

Type 5: Dealing with Sn^4: This technique is valid for Σn^2 and Σn^3. In this type, there is a series in which each term is composed of factors in an AP, i.e. factors of several terms being in AP.

$$T_n = \frac{1}{5}[(n+1)(n+2)(n+3)(n+4)[n-(n-1)]] = \frac{1}{5}(n(n+1)(n+2)(n+3)(n+4)-(n-1)(n+1)(n+2)(n+3)(n+4))$$

$$T_1 = \frac{1}{5}(1\cdot2\cdot3\cdot4\cdot5-0)$$

$$T_2 = \frac{1}{5}(2\cdot3\cdot4\cdot5\cdot6-1\cdot2\cdot3\cdot4\cdot5)$$

$$T_3 = \frac{1}{5}(3\cdot4\cdot5\cdot6\cdot7-2\cdot3\cdot4\cdot5\cdot6)$$

$$T_n = = \frac{1}{5}(n(n+1)(n+2)(n+3)(n+4)-(n-1)(n+1)(n+2)(n+3)(n+4))$$

Adding all, we have

$$S_n = \frac{1}{5}\left(n(n+1)(n+2)(n+3)(n+4)\right)$$

Note: This method will be applicable only when the series looks like the following:

a(a + d) (a+ 2d) + (a + d) (a + 2d) (a + 3d) + (a + 2d) (a + 3d) (a + 4d)+....+ up to n term, where a = first term and d = common difference

MASTERJEE CONCEPTS

- $$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+....+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

  $$\frac{1}{1.2.3}+\frac{1}{2.3.4}+....+\frac{1}{n(n+1)(n+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$$

  $$\frac{1}{a_1a_2....a_r}+\frac{1}{a_2a_3....a_{r+1}}+.....+\frac{1}{a_na_{n+1}....a_{n+r-1}}=\frac{1}{(r-1)(a_2-a_1)}\left[\frac{1}{a_1a_2....a_{r-1}}-\frac{1}{a_{n+1}a_{n+2}....a_{n+r-1}}\right]$$

- $$a_1a_2....a_r + a_2a_3....a_{r+1} +.....+ a_na_{n+1}....a_{n+r-1} = \frac{1}{(r+1)(a_2-a_1)}[a_na_{n+1}....a_{n+r} - a_0a_1a_2....a_n]$$

  Where $a_1a_2....a_n$ are in AP and $a_0 = a_1 - d$

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8. HARMONIC PROGRESSION

A sequence will be in harmonic progression (HP) if the reciprocals of its terms are in AP, e.g. if a_1, a_2, a_3, ... are in HP, then $\frac{1}{a_1},\frac{1}{a_2},\frac{1}{a_3}$ .... are in AP. For every AP, there will be a corresponding HP, and the standard H.P. is $\frac{1}{a},\frac{1}{a+d},\frac{1}{a+2d}+...........+\frac{1}{a+(n-1)d}$.

The terms of a harmonic series are the outcomes of an AP.

Note:

(i) 0 cannot be a term of H.P. because ∞ is not a term of AP, but ∞ can be a term of HP.

(ii) There is no general formula for finding the sum to n terms of HP.

(iii) If a, b and c are in HP, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP.

$$\therefore \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \quad \Rightarrow \quad b = \frac{2ac}{a+c}$$

$\Rightarrow$ a, b and c are in HP

Moreover, $\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$ i.e. $\frac{a-b}{ab} = \frac{b-c}{bc}$; i.e. $\frac{a}{c} = \frac{a-b}{b-c}$

Illustration 39: If the 3rd, 6th and last terms of a H.P. are $\frac{1}{3}, \frac{1}{5}, \frac{3}{203}$, then find the number of terms. **(JEE MAIN)**

Sol: If n^{th} term of a H.P. is $\frac{1}{a}$, then the n^{th} term of the corresponding AP will be a. Thus, by using $T_n = a + (n - 1)d$, we will get the result.

Let a be the first term and d be the common difference of the corresponding AP.

If the 3rd term of H.P. = $\frac{1}{3}$; then the 3rd term of the corresponding AP = 3

$\Rightarrow \quad a + 2d = 3$ ..... (i)

If the 6th term of H.P. = $\frac{1}{5}$; then the 6th term of the corresponding AP = 5

$\Rightarrow \quad a + 5d = 5$ ..... (ii)

From (i) and (ii), we get $d = \frac{2}{3} \Rightarrow a = \frac{5}{3}$

If the n^{th} term of H.P. = $\frac{3}{203}$; then n^{th} term of AP = $\frac{203}{3}$

$$a + (n-1)d = \frac{203}{3}; \qquad \frac{5}{3} + (n-1)\frac{2}{3} = \frac{203}{3}$$

$5 + 2n - 2 = 203; \quad n = 100$

Illustration 40: If $a_1, a_2, \ldots\ldots a_n$ are in H.P. then the expression $a_1a_2 + a_2a_3 + \ldots\ldots + a_{n-1}a_n$ is equal to.
(JEE ADVANCED)

Sol: As $\frac{1}{a_1}, \frac{1}{a_2}, \ldots\ldots \frac{1}{a_n}$ are in AP, taking $\frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_3} - \frac{1}{a_2} = \ldots \frac{1}{a_n} - \frac{1}{a_{n-1}} = d$, we can obtain the values of a_1a_2, a_2a_3 and so on.

$a_1, a_2, \ldots\ldots a_n$ are in HP

$\frac{1}{a_1}, \frac{1}{a_2}, \ldots\ldots \frac{1}{a_n}$ are in AP

$$\Rightarrow \quad \frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_3} - \frac{1}{a_2} = \ldots \frac{1}{a_n} - \frac{1}{a_{n-1}} = d \text{ (say)}$$

$$\Rightarrow \quad a_1a_2 = \frac{1}{d}(a_1 - a_2),\ a_2a_3 = \frac{1}{d}(a_2 - a_3), \ldots\ldots, a_{n-1}a_n = \frac{1}{d}(a_{n-1} - a_n)$$

Hence, $a_1a_2 + a_2a_3 + ...+a_{n-1}a_n = \frac{1}{d}[a_1 - a_2 + a_2 - a_3 + ...+a_{n-1} - a_n] = \frac{1}{d}(a_1 - a_n)$

But $\frac{1}{a_n} = \frac{1}{a_1} + (n-1)d \Rightarrow \frac{a_1 - a_n}{a_na_1} = (n-1)d$

$\therefore \quad a_1a_2 + a_2a_3 + ... + a_{n-1}a_n = (n-1)a_1a_n$

8.1 Harmonic Mean

If a, b and c are in HP, then the middle term is called the harmonic mean (HM) between them. If H is the HM between a and b, then a, H, c are in H.P. and $H = \frac{2ac}{a+c}$.

To Insert n HMs Between a and b

Let $H_1, H_2, ..., H_n$ be the n HMs between a and b.

Thus, a, $H_1, H_2, ..., H_n$ b are in HP.

$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2} .... \frac{1}{H_n}, \frac{1}{b}$ are in AP.

$\Rightarrow \frac{1}{b} = \frac{1}{a} + (n+1)d; \frac{1}{b} - \frac{1}{a} = (n+1)d; d = \frac{a-b}{ab(n+1)}$

$\Rightarrow \frac{1}{H_1} = \frac{1}{a} + d \qquad \Rightarrow \frac{1}{H_2} = \frac{1}{a} + 2d$

$\Rightarrow \frac{1}{H_3} = \frac{1}{a} + 3d \qquad \Rightarrow \frac{1}{H_n} = \frac{1}{a} + nd$

Adding all, we get

$$\sum_{i=1}^{n} \frac{1}{H_i} = \frac{n}{a} + \frac{d(n)(n+1)}{2} = \frac{n}{a} + \frac{n(n+1)}{2}\frac{(a-b)}{ab(n+1)} = n\left[\frac{1}{a} + \frac{a-b}{2ab}\right] = \frac{n}{2ab}[2b + a - b] = \frac{n(a+b)}{2ab} = n\frac{1}{H}$$

Note: The sum of the reciprocals of all the n HMs between a and b is equal to n times the reciprocal of the single HM between a and b.

For example, between 1 and $\frac{1}{100}$ if 100 HMs are inserted, then $\sum_{i=1}^{100} \frac{1}{H_i} = 5050$.

8.2 Sum of the Reciprocal of 'n' Harmonic Means

The sum of reciprocal of n harmonic means $= \frac{n(a+b)}{2ab}$

To Insert n Harmonic Means Between a and b

a, H_1, H_2, H_3 ....H_n, b $\rightarrow$ H.P.

$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3} .... \frac{1}{H_n}, \frac{1}{b} \rightarrow$ A.P.

$\frac{1}{b} = \frac{1}{a} + (n+1)d \Rightarrow (n+1)d = \frac{a-b}{ab}$

$d = \dfrac{a-b}{(n+1)ab}$

Illustration 41: Find the sum of $\dfrac{1}{H_1}+\dfrac{1}{H_2}+\dfrac{1}{H_3}.........\dfrac{1}{H_n}$.

Sol: Using $\dfrac{1}{H_n}=\dfrac{1}{a}+nd$, we can obtain the values of $\dfrac{1}{H_1}$, $\dfrac{1}{H_2}$ and so on. Then, by obtaining the value of $\sum_{n=1}^{n}\dfrac{1}{H_n}$, we will get the result.

$$\frac{1}{H_1}=\frac{1}{a}+d \qquad \frac{1}{H_2}=\frac{1}{a}+2d$$

$$\frac{1}{H_n}=\frac{1}{a}+nd \quad \Rightarrow \quad \sum_{n=1}^{n}\frac{1}{H_i}=\frac{n}{a}+\frac{n(n+1)}{2}d$$

$$= \frac{n}{a}+\frac{n(n+1)\times(a-b)}{2(n+1)ab} = \frac{n}{a}\frac{(2b+a-b)}{2b} = \frac{n}{2ab}(a+b)$$

(i) For 3 numbers a, b and c, HM is defined as the reciprocals of the mean of the reciprocals of a, b and c, i.e. means of reciprocal = $\dfrac{1}{3}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$; HM = $\dfrac{3}{\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)}$

(ii) If a_1, a_2, a_3, ..., a_n are n numbers, then

$$AM = \left(\left|\frac{a_1+a_2+a_3+a_3+........+a_n}{n}\right|\right)$$

$$GM = (a_1\, a_2\, a_3\, ...a_n)^{1/n}$$

$$HM = \left(\frac{n}{\dfrac{1}{a_1}+\dfrac{1}{a_2}+\dfrac{1}{a_2}+.......\dfrac{1}{a_n}}\right)$$

Illustration 42: If a^2, b^2, c^2 are in AP, then show that b + c, c + a, a + b are in HP.

Sol: Given that a^2, b^2 and c^2 are in AP. Thus, by adding ab + ac + bc to each term and then dividing each term by (a + b)(b + c)(c + a), we will get the result.

By adding ab + ac + bc to each term, we find that $a^2 + ab + ac + bc$, $b^2 + ba + bc + ac$, $c^2 + ca + cb + ab$ are in AP, i.e.

(a + b)(a + c), (b + c)(b + a), (c + a)(c + b) are in AP

$\therefore$ Dividing each terms by (a + b)(b + c)(c + a), we find that

$\dfrac{1}{b+c},\dfrac{1}{c+a},\dfrac{1}{a+b}$ are in AP, i.e.

b + c, c + a, a + b are in HP

Illustration 43: If H_1, H_2, ..., H_n are n harmonic means between a and b ($\neq$ a), then find the value of $\dfrac{H_1+a}{H_1-a}+\dfrac{H_n+b}{H_n-b}$.

Sol: As a, H_1, H_2, ..., H_n, b are in HP, $\dfrac{1}{a},\dfrac{1}{H_1},\dfrac{1}{H_2}.......\dfrac{1}{H_n},\dfrac{1}{b}$ are in AP. By considering d as the common difference of this

AP and using T_n = a + (n – 1) d we can solve this problem.

$\frac{1}{b} = \frac{1}{a} + (n + 1)d$ and $\frac{1}{H_n} - \frac{1}{H_1} = (n - 1)d$

Now, $$\frac{H_1 + a}{H_1 - a} = \frac{1/a + 1/H_1}{1/a - 1/H_1} = \frac{1/a + 1/H_1}{-d}$$

and $$\frac{H_n + b}{H_n - b} = \frac{1/b + 1/H_n}{1/b - 1/H_n} = \frac{1/b + 1/H_n}{d}$$

$$\therefore \quad \frac{H_1 + a}{H_1 - a} + \frac{H_n + b}{H_n - b} = \frac{1/a + 1/H_1}{-d} + \frac{1/b + 1/H_n}{d} = \frac{1}{d}\left[\left(\frac{1}{b} - \frac{1}{a}\right) + \left(\frac{1}{H_n} - \frac{1}{H_1}\right)\right] = 2n$$

9. RELATION BETWEEN AM, G.M. AND HM

If a and b are two positive numbers, then it can be shown that $A \geq G \geq H$ and A, G, H are in GP, i.e. $G^2 = AH$.

Proof: Given that, $A = \frac{a+b}{2}$, $G = \sqrt{ab}$ and $H = \frac{2ab}{a+b}$

$$\therefore \quad A - G = \frac{a+b}{2} - \sqrt{ab}$$

$$\Rightarrow \quad A - G = \frac{(\sqrt{a} - \sqrt{b})^2}{2} \geq 0$$

$$\Rightarrow \quad A \geq G \qquad \text{......(i)}$$

$$G - H = \sqrt{ab} - \frac{2ab}{a+b}$$

$$\Rightarrow \quad G - H = \sqrt{ab}\left(\frac{a + b - 2\sqrt{ab}}{a+b}\right) \quad \Rightarrow \quad G - H = \frac{\sqrt{ab}}{a+b}(\sqrt{a} - \sqrt{b})^2 \geq 0$$

$$\Rightarrow \quad G \geq H \qquad \text{......(ii)}$$

Using (i) and (ii), we find that

$$A \geq G \geq H$$

Please note that the equality holds only when a = b.

Proof of G^2 = AH

Proof: $A = \frac{a+b}{2}$, $G = \sqrt{ab}$ and $H = \frac{2ab}{a+b}$

Now, $AH = ab = G^2$ $\Rightarrow$ A, G & H are in G.P.

Moreover, $\frac{A}{G} = \frac{G}{H}$; $\therefore$ $A \geq G$ $\Rightarrow$ $G \geq H$

Therefore, $A \geq G \geq H$; in fact, RMS ≥ A.M. ≥ G.M. ≥ HM (where RMS is root mean square).

MASTERJEE CONCEPTS

- If a and b are two positive quantities, then AM, G.M. and HM are always in GP, i.e. only for two numbers.
- If there are three numbers. then AM, G.M. and HM are in G.P. only when the three numbers. are in GP.

  For example, 2, 4, 8 $\rightarrow$ GP

  GM = 4; A.M. = $\frac{14}{3}$; HM = $\frac{24}{7}$
- For two positive numbers, it has been shown that $A \geq G \geq H$, equality holding for equal numbers.
- For n non-zero positive numbers, it has been shown that $A \geq G \geq H$, equality holding when all the numbers are equal.

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Illustration 44: If a, b and c are unequal positive numbers in HP, then prove that

$$\frac{a+b}{2a-b}+\frac{c+b}{2c-b} > 4.$$

(JEE ADVANCED)

Sol: As a, b and c are in HP, therefore $\frac{2}{b}=\frac{1}{a}+\frac{1}{c}$. Thus, by substituting this to LHS, we can prove the given problem.

$$\text{LHS} = \frac{\frac{1}{b}+\frac{1}{a}}{\frac{2}{b}-\frac{1}{a}}+\frac{\frac{1}{b}+\frac{1}{c}}{\frac{2}{b}-\frac{1}{c}}=\frac{\frac{1}{b}+\frac{1}{a}}{\frac{1}{c}}+\frac{\frac{1}{b}+\frac{1}{c}}{\frac{1}{a}},\ \text{using (i)}$$

$$=\frac{c}{b}+\frac{c}{a}+\frac{a}{b}+\frac{a}{c}=\frac{a+c}{b}+\frac{a}{c}+\frac{c}{a}.$$

Now, A.M. > G.M. $\Rightarrow \frac{\frac{a}{c}+\frac{c}{a}}{2} > \sqrt{\frac{a}{c}.\frac{c}{a}}$ or $\frac{a}{c}+\frac{c}{a} > 2.$

$$\frac{a+c}{b}=\frac{a+c}{\frac{2ac}{a+c}}=\frac{(a+c)^2}{2ac}=\frac{(a-c)^2}{2ac}+2$$

$$\therefore \text{LHS} = \frac{a+c}{b}\left(\frac{a}{c}+\frac{c}{a}\right) > 2 + 2 = 4$$

10. PROPERTIES OF AM, G.M. AND HM

(i) The equation with a and b as its roots is $x^2 - 2Ax + G^2 = 0$

Proof: The equation with a and b as its roots is $x^2 - (a + b)\,x + ab = 0$

$\Rightarrow \quad x^2 - 2Ax + G^2 = 0 \qquad (\because A = \frac{a+b}{2}, G = \sqrt{ab})$

(ii) If A, G and H are the arithmetic, geometric and harmonic means, respectively, between three given numbers a, b and c, then the equation having a, b, c as its roots is $x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$

Proof: As given, A = $\frac{a+b+c}{3}$, G = $(abc)^{1/3}$ and $\frac{1}{H} = \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}$

$\Rightarrow$ a + b + c = 3A, abc = G^3 and $\frac{3G^3}{H}$ = ab + bc + ca

The equation having a, b and c as its roots is $x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc = 0$

$\Rightarrow \quad x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$

Illustration 45: The harmonic means between two numbers is given as 4, their A.M. is A, and G.M. is G, satisfy the relation $2A + G^2 = 27$. Determine the two numbers. **(JEE ADVANCED)**

Sol: Let a and b be the two numbers and H = 4 be the harmonic mean between them. Therefore, by using A.M. = $\frac{a+b}{2}$ and G.M. = $\sqrt{ab}$, we can obtain the values of a and b.

H = 4 (given)

As A, G and H are in GP, therefore $G^2 = AH \Rightarrow \quad G^2 = 4A$

Also, $2A + G^2 = 27$ (given; $\therefore G^2 = 4A$)

$\therefore \quad 6A = 27$

$\Rightarrow \quad A = \frac{9}{2} \qquad \Rightarrow \quad \frac{a+b}{2} = \frac{9}{2} \Rightarrow \quad a + b = 9$

We have, $G^2 = 4A$ and $A = 9/2 \Rightarrow \quad G^2 = 18 \quad \Rightarrow \quad ab = 18$

The quadratic equation having a and b as its roots is $x^2 - (a + b)x + ab = 0$ or, $x^2 - 9x + 18 = 0$

$\Rightarrow x = 3, 6$

Thus, the two numbers are 3 and 6.

Illustration 46: If 2a + b + 3c = 1 and a > 0, b > 0, c > 0, then find the greatest value of $a^4b^2c^2$ and obtain the corresponding values of a, b and c. **(JEE ADVANCED)**

Sol: Since there is a^4,. take four equal parts of 2a; as there is b^2, take two equal parts of b; as there is c^2, take two equal parts of 3c. Since A.M. ≥ G.M., obtaining A.M. and G.M. of these numbers will help in solving this illustration.

Let us consider the positive numbers $\frac{2a}{4}, \frac{2a}{4}, \frac{2a}{4}, \frac{2a}{4}, \frac{b}{2}, \frac{b}{2}, \frac{3c}{2}, \frac{3c}{2}$.

For the numbers, $A = \frac{\frac{2a}{4}+\frac{2a}{4}+\frac{2a}{4}+\frac{2a}{4}+\frac{b}{2}+\frac{b}{2}+\frac{3c}{2}+\frac{3c}{2}}{4+2+2} = \frac{2a+b+3c}{8} = \frac{1}{8}$

($\therefore$ 2a + b + 3c = 1)

$$G = \left(\frac{2a}{4}.\frac{2a}{4}.\frac{2a}{4}.\frac{2a}{4}.\frac{b}{2}.\frac{b}{2}.\frac{3c}{2}.\frac{3c}{2}\right)^{\frac{1}{8}} = \left(\frac{1}{2^4}.\frac{1}{2^2}.\frac{1}{2^2}.3^2a^4b^2c^2\right)^{\frac{1}{8}}$$

$$\therefore \quad A \geq G \quad \Rightarrow \quad \frac{1}{8} \geq \left(\frac{3^2}{2^8}a^4b^2c^2\right)^{\frac{1}{8}}$$

or $\frac{1}{8^8} \geq \frac{3^2}{2^8}a^4b^2c^2$ or $\frac{2^8}{3^2.8^8} \geq a^4b^2c^2$ or $\frac{1}{9.4^8} \geq a^4b^2c^2$.

Hence, the greatest value of $a^4b^2c^2 = \dfrac{1}{9.4^8}$

It has been found that when the equality holds, the greatest value takes place.

We know that A = G when all the numbers are equal, i.e.

$$\frac{2a}{4} = \frac{b}{2} = \frac{3c}{2} \quad \Rightarrow \quad a = b = 3c$$

$$\therefore \quad \frac{a}{3} = \frac{b}{3} = \frac{c}{1} = k \quad \therefore \quad a = 3^k,\ b = 3k,\ c = k$$

$$\therefore \quad 2a + b + 3c = 1 \quad \Rightarrow \quad 6k + 3k + 3k = 1$$

$$\therefore \quad k = \frac{1}{12} \quad \therefore \quad a = \frac{3}{12},\ b = \frac{3}{12},\ c = \frac{1}{12},\ \text{i.e. } a = \frac{1}{4},\ b = \frac{1}{4},\ c = \frac{1}{12}$$

Arithmetic Mean of the mth power

Suppose $a_1, a_2, \ldots, a_n$ be n positive real numbers (not all equal) and let m be a real number, then

$$\frac{a_1^m + a_2^m + \ldots a_n^m}{n} > \left(\frac{a_1 + a_2 + \ldots a_n}{n}\right)^m, \text{ if } m \in R - [0, 1]$$

If $m \in (0, 1)$, then $\dfrac{a_1^m + a_2^m + \ldots a_n^m}{n} < \left(\dfrac{a_1 + a_2 + \ldots a_n}{n}\right)^m$

Thus, if $m \in \{0, 1\}$, then $\dfrac{a_1^m + a_2^m + \ldots a_n^m}{n} = \left(\dfrac{a_1 + a_2 + \ldots a_n}{n}\right)^m$

PROBLEM-SOLVING TACTICS

(a) When looking for a pattern in a sequence or series, writing out several terms will help you see the pattern, do not simplify directly. If you do this way, it is often easier to spot the pattern (if you leave terms as products, sums, etc.).

(b) If each term of an AP is multiplied by (or divided by a non-zero) fixed constant C, the resulting sequence is also an AP, with a common difference C times $\left(\text{or } \dfrac{1}{c} \text{times}\right)$ the previous.

(c) Tips for AP problems

(i) When the number of terms are three, then we take the terms as a – d, a, a + d;

Five terms as a - 2d, a - d, a, a + d, a + 2d

Here, we take the middle term as ‘a’ and common difference as ‘d’.

(ii) When the number of terms is even, then we take:

Four terms as a – 3d, a – d, a + d, a + 3d;

Six terms as a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d

Here, we take ‘a – d’ and ‘a + d’ as the middle terms and common difference as ‘2d’.

(iii) If the number of terms in an AP is even, then take the number of terms as 2n and if odd then take it as (2n + 1).

(d) Tips for G.P. problems

(i) When the number of terms is odd, then we take three terms as a/r, a, ar; five terms as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$. Here, we take the middle term as 'a' and common ratio as 'r.'

(ii) When the number of terms is even, then we take four terms as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$; six terms as $\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$. Here, we take $\frac{'a'}{r}$ and 'ar' as the middle terms and common ratio as 'r^2.'

(e) Tips for H.P. problems

For three terms, we take as $\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$

For four terms, we take as $\frac{1}{a-3d}, \frac{1}{a-d}, \frac{1}{a+d}, \frac{1}{a+3d}$

For five terms, we take as $\frac{1}{a-2d}, \frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}$

FORMULAE SHEET

Arithmetic Progression: Here, a, d, A and S_n represent the first term, common difference, A.M. and sum of the numbers, respectively, and T_n stands for the n^{th} term.

1.	$T_n = a + (n-1)d$	4.	$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$
2.	$T_n = \frac{T_{n-1}+T_{n+1}}{2}$	5.	$A = \frac{(a_1 + a_2 + \ldots + a_n)}{n}$
3.	$S_n = \frac{n}{2}(a+T_n)$	6.	Insertion of n arithmetic means between a and b is $A_n = a + \frac{n(b-a)}{n+1}$

Geometric Progression: Here, a, r, S_n and G represent the first term, common ratio, sum of the terms and G.M., respectively, and T_n stands for the n^{th} term.

1.	$T_n = a.r^{n-1}$	4.	$S_n = \frac{a(r^n-1)}{r-1}$
2.	$T_n = \sqrt{T_{n-1}\cdot T_{n+1}}$	5.	$S_\infty = \frac{a}{1-r}$ (for $-1 < r < 1$)
3.	$S_n = \frac{T_{n+1}-a}{r-1}$	6.	Insertion of n geometric means between a and b is $G_1 = ar$, $G_2 = ar^2$ ....$G_n = ar^n$ or $G_n = b/r$, where $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Arithmetic Geometric Progression: Here, a = the first term of AP, b = the first term of GP, d = common difference and r = common ratio of GP.

1.	$S_n = ab + (a + d)br + (a + 2d)br^2 + (a + 3d)br^3 + ...$
2.	$S_n = \frac{ab}{1-r} + \frac{dbr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]br^n}{1-r}$
3.	$S_\infty = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$ (for $-1 < r < 1$)

Harmonic Progression

1. $a_n = \frac{1}{a+(n-1)d}$, where $a = \frac{1}{a_1}$ and $d = \frac{1}{a_2} - \frac{1}{a_1}$

2. $\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right)$

3. Insertion of n harmonic means between a and b

$$\frac{1}{H_1} = \frac{1}{a} + \frac{a-b}{(n+1)ab}$$

$$\frac{1}{H_2} = \frac{1}{a} + \frac{2(a-b)}{(n+1)ab} \text{ and so on } \Rightarrow \left[\frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{(n+1)ab}\right]$$

1.	The sum of n natural numbers	$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$
2.	The sum of n odd natural numbers	$\sum_{r=1}^{n} (2r-1) = n^2$
3.	The sum of n even natural numbers	$\sum_{r=1}^{n} 2r = n(n+1)$
4.	The sum of squares of n natural numbers	$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$
5.	The sum of cubes of n natural numbers	$\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$