2. QUADRATIC EQUATIONS AND INEQUALITIES

1. QUADRATIC POLYNOMIAL

Quadratic polynomial: A Polynomial of degree 2 in one variable of the type $f(x) = ax^2 + bx + c$ where a, b, c, $\in R$ and $a \neq 0$ is called a quadratic polynomial. 'a' is called the leading coefficient and 'c' is called the absolute term of f (x). If a = 0, then y = bx + c is called a linear polynomial and if $a = 0, b \neq 0$ & c = 0 then y = bx is called an odd linear polynomial since $f(y) + f(-y) = 0$

Standard appearance of a polynomial of degree n is $f(x) = a_n X^n + a_{n-1} X^{x-1} + a_{n-2} X^{n-2} + ... + a_1 X + a_0$

Where $a_n \neq 0$ & a_n, $a_{n-1}, ... a_0 \in R$; n = 0, 1, 2...

When the Highest exponent is 3 $\rightarrow$ It is a cubic polynomial

When the Highest exponent 4 $\rightarrow$ It is a biquadratic polynomial

For different values of a, b, and c there can be 6 different graphs of $y = ax^2 + bx + c$

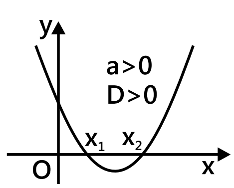
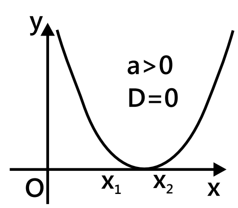


Figure 2.1



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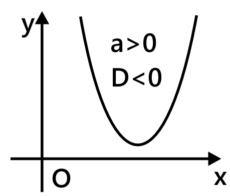

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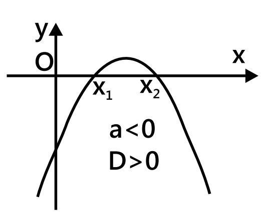

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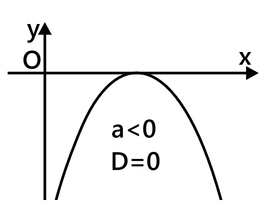

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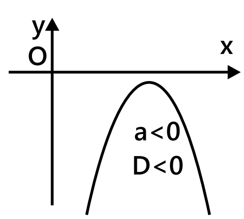


Figure 2.6

Figure 1 and figure 4 $\Rightarrow x_1, x_2$ are the zeros of the polynomial

Figure 2 $\Rightarrow$ zeros of the polynomial coincide, i.e $ax^2 + bc + c$ is a perfect square; $y \geq 0 \ \forall \ x \in R$

Figure 3 $\Rightarrow$ polynomial has no real zeros, i.e the quantity $ax^2 + bx + c > 0$ for every $x \in R$

Figure 5 $\Rightarrow$ zeros of the polynomial coincide, i.e $ax^2 + bx + c$ is a perfect square; $y \leq 0 \ \forall \ x \in R$

Figure 6 $\Rightarrow$ polynomial has no real zeros; $ax^2 + bx + c < 0, \forall \ x \in R \ < 0$

2. QUADRATIC EQUATION

A quadratic polynomial expression equated to zero becomes a quadratic equation and the values of x which satisfy the equation are called roots/ zeros of the Quadratic Equation.

General form: $ax^2 + bx + c = 0$

Where a, b, c, $\in$ R and $a \neq 0$, the numbers a, b and c are called the coefficients of the equation.

a is called the leading coefficient, b is called the middle coefficient and c is called the constant term.

e.g $3x^2 + x + 5 = 0, -x^2 + 7x + 5 = 0, x^2 + x = 0, x^2 = 0$

2.1 Roots of an Equation

The values of variables satisfying the given equation are called its roots.

In other words, x = α is a root of the equation f(x), if $f(\alpha) = 0$. The real roots of an equation f(x) = 0 are the x-coordinates of the points where the curve y = f(x) intersect the x-axis. e.g. $x^2 - 3x + 2 = 0$. At x = 1 & 2 the equation becomes zero.

Note: A Polynomial can be rewritten as given below

$y = a(x - r_1)(x - r_2)....(x - r_n)$

The factors like $(x - r_1)$ are called linear factors, because they describe a line when you plot them.

2.2 Dividing Polynomials

Dividing polynomials: When 13 is divided by 5, we get a quotient 2 and a remainder 3.

Another way to represent this example is : 13 = 2 × 5 + 3

The division of polynomials is similar to this numerical example. If we divide a polynomial by (x – r), we obtain a result of the form:

F(x) =(x – r) q(x) + R, where q(x) is the quotient and R is the remainder.

Illustration 1: Divide $3x^2 + 5x - 8$ by (x – 2) **(JEE MAIN)**

Sol: Similar to division of numbers, we can write the given

$$\begin{array}{r} 3x+11 \\ x-2\overline{)3x^2+5x-8} \\ \underline{3x^2-6x} \\ 11x-8 \\ \underline{11x-22} \\ 14 \end{array}$$

polynomial as $3x^2 + 5x - 8 = (x - 2)q(x) + R$.

Thus, we can conclude that $3x^2 + 5x - 8 = (x - 2)(3x + 11) + 14$

Where the quotient $q(x) = 3x + 11$ and the remainder R = 14.

3. REMAINDER AND FACTOR THEOREM

3.1 Remainder Theorem

Consider $f(x) = (x - r)q(x) + R$

Note that if we take x = r, the expression becomes

$$f(r) = (r - r)q(r) + R\,; \qquad \Rightarrow f(r) = R$$

This leads us to the Remainder Theorem which states:

If a polynomial $f(x)$ is divided by (x – r) and a remainder R is obtained, then $f(r) = R$.

Illustration 2: Use the remainder theorem to find the remainder when $f(x) = 3x^2 + 5x - 8$ is divided by (x – 2) **(JEE MAIN)**

Sol: Use Remainder theorem. Put x = 2 in f(x). Since we are dividing $f(x) = 3x^2 + 5x - 8$ by (x – 2), we consider x = 2. Hence, the remainder R is given by

$R = f(2) = 3\,(2)^2 + 5(2) - 8 = 14$

This is the same remainder we arrived at with the preceding method.

Illustration 3: By using the remainder theorem, determine the remainder when $3(x)^3 - x^2 - 20x + 5$ is divided by (x + 4) **(JEE MAIN)**

Sol: As in Illustration 2, we can solve this problem by taking r = - 4

f(x) = $3(x)^3 - x^2 - 20x + 5$.

Therefore the remainder R = $f(-4) = 3(-4)^3 - (-4)^2 - 20(-4) + 5 = -192 - 16 + 80 + 5 = -123$

3.2 Factor Theorem

The Factor Theorem states:

If the remainder $f(r) = R = 0$, then (x – r) is a factor of $f(x)$.

The Factor Theorem is powerful because it can be used to calculate the roots of polynomial equations having degree more than 2.

Illustration 4: Find the remainder R by long division and by the Remainder Theorem $\left(2x^4 - 10x^2 + 30x - 60\right) \div (x + 4)$. **(JEE MAIN)**

Sol: We can find the remainder in the given division problem by using the long division method, i.e. similar to number division and also by the Remainder theorem, i.e. R = f(r).

Now using the Remainder Theorem: $f(x) = 2x^4 - 10x^2 + 30x - 60$

Remainder $= f(-4) = 2(-4)^4 - 10(-4)^2 + 30(-4) - 60 = 172$

This is the same answer we achieved by the long division method.

$$2x^3 - 8x^2 + 22x - 58$$
$$x + 4 \,\overline{)\; 2x^4 + 0x^3 - 10x^2 + 30x - 60}$$
$$\boxed{-2x^4 - 8x^4}$$
$$-8x^3 - 10x^2$$
$$\boxed{8x^3 + 32x^2}$$
$$22x^2 + 30x$$
$$\boxed{-22x^2 - 88x}$$
$$-58x - 60$$
$$\boxed{58x + 232}$$
$$172$$

Illustration 5: Use the factor theorem to decide if (x – 2) is a factor of $f(x) = x^5 - 2x^4 + 3x^3 - 6x^2 - 4x + 8$.
(JEE MAIN)

Sol: We know that (x – r) will be a factor of f(x) if f(r) = 0. Therefore, by using this condition we can decide whether (x – 2) is a factor of the given polynomial or not.

$$f(x) = x^5 - 2x^4 + 3x^3 - 6x^2 - 4x + 8$$

$$f(2) = (2)^5 - 2(2)^4 + 3(2)^3 - 6(2)^2 - 4(2) + 8 = 0$$

Since $f(2) = 0$, we can conclude that (x – 2) is a factor.

Illustration 6: If x is a real number such that $x^3 + 4x = 8$, then find the value of the expression $x^7 + 64x^2$.
(JEE MAIN)

Sol: Represent $x^7 + 64x^2$ as a product of $x^3 + 4x - 8 = 0$ and some other polynomial + constant term. The value of the expression will be equal to the constant term.

Given $x^3 + 4x - 8 = 0$;

Now $y = x^7 + 64x^2 = x^4(x^3 + 4x - 8) - 4x^5 + 8x^4 + 64x^2$

$= -4x^2\left(x^3 + 4x - 8\right) + 16x^3 + 64x = 16\left(x^3 + 4x - 8\right) + 128 = 128$

Alter: $x^3 + 4x = 8$

Now divide $x^7 + 64x^2$ by $x^3 + 4x - 8 \Rightarrow \dfrac{x^7 + 64x^2}{x^3 + 4x - 8}$

Here, after division, the remainder will be the value of the expression $x^7 + 64x^2$.

Thus, after dividing, the value is 128.

Illustration 7: A cubic polynomial P(x) contains only terms of the odd degree. When P (x) is divided by (x – 3), then the remainder is 6. If P(x) is divided by $\left(x^2 - 9\right)$, then the remainder is g(x). Find the value of g(2). **(JEE MAIN)**

Sol: Let p(x) = $ax^3 + bx$, and use Remainder theorem to get the value of g(2).

Let p(x) = $ax^3 + bx$; By remainder theorem P(3) =6

P(3) = 3(b + 9a) = 6 ; 9a + b = 2 .....(i)

P(x) = $\left(x^2 - 9\right)$ ax + (b + 9a) x

Given that the remainder is g(x) when P(x) is divided by $\left(x^2 - 9\right)$

$\therefore$ g(x)=(b+9a) x

From (i) (b+9a) = 2 $\quad \therefore g(x) = 2x \quad \therefore g(2) = 4$

4. METHODS OF SOLVING QUADRATIC EQUATIONS

There are two methods to solve a Quadratic equation

(i) Graphical (absolute) (ii) Algebraic

Algebraic Method

$ax^2 + bx + c = 0$; Divide by a

$x^2 + \dfrac{bx}{a} + \dfrac{c}{a} = 0 \qquad \Rightarrow \left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2}{4a^2} - \dfrac{c}{a} = \dfrac{b^2 - 4ac}{4a^2}$

$x+\frac{b}{2\,a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$; $\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$b^2-4ac=D$ (Discriminant)

$\alpha=\frac{-b+\sqrt{D}}{2a}$, $\beta=\frac{-b-\sqrt{D}}{2a}$; $\alpha+\beta=\frac{-b}{a}$, $\alpha.\beta=\frac{c}{a}$

$ax^2+bx+c=0\Rightarrow x^2-(\alpha+\beta)x+(\alpha.\beta)=0$

5. NATURE OF ROOTS

Given the Quadratic Equation $ax^2+bx+c=0$, where a, b, c, $\in$ R and a $\neq$ 0

Discriminant: D = b^2-4ac

D < 0	D = 0	D > 0	
Roots are imaginary & are given by $\alpha+i\,\beta,\ \alpha-i\beta$	Roots are real and equal and are given by -b/2a.	D is a perfect square then roots are rational and different, provided a, b, c, $\in$ Q	D is not a perfect square then roots are real and distinct and are of the form P + $\sqrt{q}$ & p - $\sqrt{q}$, provided a, b, c, $\in$ Q

For the quadratic equation $ax^2+bx+c=0$ ...(i)

(i) If a, b, c, $\in$R and a $\neq$ 0, then

(a) If D < 0, then equation (i) has non-real complex roots.

(b) If D > 0, then equation (i) has real and distinct roots, namely

$$\alpha=\frac{-b+\sqrt{D}}{2a},\beta=\frac{-b-\sqrt{D}}{2a}$$

And then $ax^2+bx+c=a(x-\alpha)(x-\beta)$ ....(ii)

(c) If D = 0, then equation (i) has real and equal roots. $\alpha=\beta=-\frac{b}{2a}$ and then $ax^2+bx+c=a(x-\alpha)^2$ ...(iii)

(ii) If a, b, c $\in$ Q and D is a perfect square of a rational number, then the roots are rational numbers, and in case D is not a perfect square then the roots are irrational.

(iii) If a, b, c $\in$ R and p + iq is one root of equation (i) (and q $\neq$0) then the other must be the conjugate p – iq and vice-versa. (p,q x^2 R and i^2 = –1).

If a, b, c $\in$ Q and p +.$\sqrt{q}$. is one root of equation (i) then the other must be the conjugate

p - $\sqrt{q}$ and vice-versa (where p is a rational and $\sqrt{q}$ is an irrational surd).

(iv) If exactly 1 root of Quadratic Equation is 0 then the product of roots = 0

$\Rightarrow$ c = 0

$\therefore$ The equation becomes $y=ax^2+bx=0$ the graph of which passes through the origin as shown in Fig 2.7.

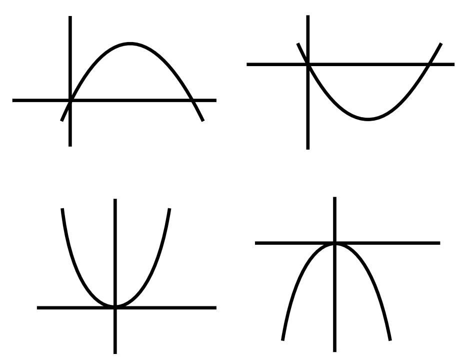


Figure 2.7

(v) If both roots of quadratic equation are 0 then S = 0 & P = 0 where S = sum and P = Product

$\Rightarrow \frac{b}{a} = 0 \;\; \& \;\; \frac{c}{a} = 0 \;\; \therefore b = c = 0 \qquad \therefore y = ax^2$

(vi) If exactly one root is infinity $\qquad ax^2 + bx + c = 0$

$x = \frac{1}{y}$, then $\frac{a}{y^2} + \frac{b}{y} + c = 0 \quad ; \quad cy^2 + by + a = 0$ must have exactly one root 0

$\therefore\; P = 0 \Rightarrow \frac{a}{c} = 0 \qquad \Rightarrow \; a = 0\,;\; c \neq 0 \qquad \Rightarrow Y = bx + c$

MASTERJEE CONCEPTS

Very important conditions

- If $y = ax^2 + bx + c$ is positive for all real values of x then a > 0 & D < 0
- If $y = ax^2 + bx + c$ is negative for all real values of x then a < 0 & D < 0
- If both roots are infinite for the equation $ax^2 + bx + c = 0$; $\quad x = \frac{1}{y} \qquad \Rightarrow \frac{a}{y^2} + \frac{b}{y} + c = 0$

$cy^2 + by + a = 0 \qquad -\frac{b}{c} = 0, \frac{a}{c} = 0 \quad \therefore a = 0, b = 0 \;\; \& \;\; c \neq 0$

Vaibhav Gupta (JEE 2009 AIR 54)

5.1 Roots in Particular Cases

For the quadratic equation $ax^2 + bx + c = 0$

(a) If b = 0, ac < 0 $\Rightarrow$ Roots are of equal magnitude but of opposite sign;

(b) If c = 0 $\Rightarrow$ One root is zero, the other is –b/a;

(c) If b = c = 0 $\Rightarrow$ Both roots are zero;

(d) If a = c $\Rightarrow$ The roots are reciprocal to each other;

(e) If $\begin{pmatrix} a > 0; c < 0 \\ a < 0;\; c > 0 \end{pmatrix}$ $\Rightarrow$ The roots are of opposite signs;

(f) If the sign of a = sign of b × sign of c $\Rightarrow$ the root of greater magnitude is negative;

(g) If a + b + c = 0 $\Rightarrow$ one root is 1 and the other is c/a;

(h) If a = b = c =0 then the equation will become an identity and will be satisfied by every value of x.

Illustration 8: Form a quadratic equation with rational coefficients having $\cos^2\frac{\pi}{8}$ as one of its roots. **(JEE MAIN)**

Sol: If the coefficients are rational, then the irrational roots occur in conjugate pairs. Hence if one root is $\left(\alpha + \sqrt{\beta}\right)$ then other one will be $\left(\alpha - \sqrt{\beta}\right)$, Therefore, by using the formula $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$ we can obtain the required equation.

$$\cos^2\frac{\pi}{8} = \frac{1}{2} \times 2\cos^2\frac{\pi}{8} = \frac{1}{2}\left(1 + \cos\frac{\pi}{4}\right) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$

Thus, the other root is $\frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$ also Sum of roots = 1 and Product of roots = $\frac{1}{8}$

Hence, the quadratic equation is $8x^2 - 8x + 1 = 0$

Illustration 9: Find the quadratic equation with rational coefficients when one root is $\frac{1}{\left(2+\sqrt{5}\right)}$. **(JEE MAIN)**

Sol: Similar to Illustration 8.

If the coefficients are rational, then the irrational roots occur in conjugate pairs. Given that if one root is

$\alpha = \frac{1}{(2+\sqrt{5})} = \sqrt{5}-2$, then the other root is $\beta = \frac{1}{(2-\sqrt{5})} = -\sqrt{5}-2$

Sum of the roots $\alpha + \beta = -4$ and product of roots $\alpha\,\beta$ = -1. Thus, the required equation is $x^2 + 4x - 1 = 0$.

Illustration 10: If $\cos\theta$, $\sin\phi$, $\sin\theta$ are in G.P then check the nature of the roots of $x^2 + 2\cot\phi\, x + 1 = 0$?
(JEE MAIN)

Sol : As $\cos\theta, \sin\phi$ and $\sin\theta$ are in G.P., so we have $\sin^2\phi = \cos\theta.\sin\theta$. By calculating the discriminant (D), we can check nature of roots.

We have $\sin^2\phi = \cos\theta\,\sin\theta\,\theta$ (as $\cos\,\theta$, $\sin\phi$, $\sin\theta$ are in GP)

D = 4 $\cot^2\phi$ - 4

$$= 4\left[\frac{\cos^2\varphi - \sin^2\phi}{\sin^2\phi}\right] = \frac{4(1-2\sin^2\phi)}{\sin^2\phi} = \frac{4(1-2\sin\theta\cos\theta)}{\sin^2\phi} = \left[\frac{2(\sin\theta-\cos\theta)}{\sin\phi}\right]^2 \ge 0$$

Hence the roots are real

Illustration 11: Form a quadratic equation with real coefficients when one root is 3 – 2i. **(JEE MAIN)**

Sol: Since the complex roots always occur in pairs, so the other root is 3 + 2i. Therefore, by obtaining the sum and the product of the roots, we can form the required quadratic equation.

The sum of the roots is

$\left(3+2i\right)+\left(3-2i\right)=6$. The product of the root is $\left(3+2i\right)\times\left(3-2\,i\right)=9-4i^2=9+4=13$

Hence, the equation is $x^2 - Sx + P = 0$

$\Rightarrow x^2 - 6x + 13 = 0$

Illustration 12: If p, q and r are positive rational numbers such that p>q>r and the quadratic equation (p + q – 2r) x^2 + (q + r – 2p)x + (r + p – 2q) = 0 has a root in (-1, 0) then find the nature of the roots of $px^2 + 2qx + r = 0$
(JEE ADVANCED)

Sol : In this problem, the sum of all coefficients is zero. Therefore one root is 1 and the other root is .

$\left(\frac{r+p-2q}{p+q-2r}\right)$. which also lies in (-1, 0). Hence, by solving $-1 < \frac{r+p-2q}{p+q-2r} < 0$ we can obtain the nature of roots of $px^2 + 2qx + r = 0$.

(p + q – 2r) x^2 + (q + r – 2p)x + (r + p – 2q) = 0

$\because$ One root is 1 & other lies in (-1, 0) $\Rightarrow$ $-1 < \frac{r+p-2q}{p+q-2r} < 0$ and p > q > r, Then p+q-2r > 0

r + p - 2q< 0 $\Rightarrow$ r + p < 2q $\Rightarrow$ $\frac{r+p}{q} < 2$

$r^2 + p^2 + 2pr < 4q^2 \Rightarrow 4pr < 4q^2 \Rightarrow q^2 > pr$ $\quad [\because\ q^2 > pr]$

Hence D > 0, so the equation $px^2 + 2qx + r = 0$ has real & distinct roots.

Illustration 13: Consider the quadratic polynomial $f(x) = x^2 - px + q$ where $f(x) = 0$ has prime roots. If p + q = 11 and $a = p^2 + q^2$, then find the value of $f(a)$ where a is an odd positive integer. **(JEE ADVANCED)**

Sol: Here f(x) = $x^2 - px + q$, hence by considering α and β as its root and using the formulae for sum and product of roots and the given conditions, we get the values of f(a).

f(x) = $x^2 - px + q$

Given α and β are prime

$\alpha + \beta = p$ ... (i);

$\alpha\beta = q$ ... (ii)

Given p + q = 11 $\quad \Rightarrow \alpha + \beta + \alpha\beta = 11$

$\Rightarrow (\alpha+1)(\beta+1) = 12$; $\alpha = 2, \beta = 3$ are the only primes that solve this equation.

$\therefore$ f(x) = (x – 2)(x – 3) = $x^2 - 5x + 6$

$\therefore$ p = 5, q = 6 $\Rightarrow$ a = $p^2 + q^2$ = 25 + 36 = 51; f(51) = (51 – 2)(51 – 3) = 49 × 48 = 3422

Illustration 14: Find the maximum vertical distance 'd' between the parabola $y = -2x^2 + 4x + 3$ and the line y = x – 2 through the bounded region in the figure. **(JEE MAIN)**

Sol: In this problem, the maximum vertical distance d means the value of y.

The vertical distance is given by

$d = -2x^2 + 4x + 3 - (x-2) = -2x^2 + 3x + 5$,

which is a parabola which opens downwards.

Its maximum value is the y-coordinate of

the vertex which has x-coordinate equal to $\frac{-b}{2a} = \frac{-3}{2(-2)} = \frac{3}{4}$.

Then $y = -2\left(\frac{3}{4}\right)^2 + 3\left(\frac{3}{4}\right) + 5 = \frac{-9}{8} + \frac{18}{8} + \frac{40}{8} = \frac{49}{8}$

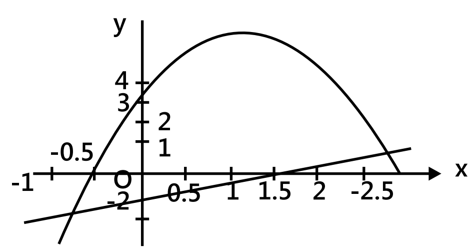


Figure 2.8

Illustration 15: $y = ax^2 + bx + c$ has no real roots. Prove that c(a+b+c) > 0. What can you say about expression c(a – b + c)? **(JEE ADVANCED)**

Sol: Since there are no real roots, y will always be either positive or negative. Therefore $f(x_1)f(x_2) > 0$

$f(0)f(1) > 0 \Rightarrow c(a+b+c) > 0$; similarly $f(0)\ f(-1) > 0 \Rightarrow c\ (a-b+c) > 0$

Illustration 16: α, β are roots of the equation $f(x) = x^2 - 2x + 5 = 0$, then form a quadratic equation whose roots are $\alpha^3 + \alpha^2 - \alpha + 22$ & $\beta^3 + 4\beta^2 - 7\beta + 35$. **(JEE MAIN)**

Sol: As α, β are roots of the equation $f(x) = x^2 - 2x + 5 = 0$, $f(\alpha)$, and f(β) will be 0. Therefore, by obtaining the values of $\alpha^3 + \alpha^2 - \alpha + 22$ and $\beta^3 + 4\beta^2 - 7\beta + 35$ we can form the required equation using sum and product method.

From the given equation $\alpha^2 - 2\alpha + 5 = 0$ and $\beta^2 - 2\beta + 5 = 0$

We find $\alpha^3+\alpha^2-\alpha+22 = \alpha(\alpha^2-2\alpha+5)+3\alpha^2-6\alpha+22 = 3(\alpha^2-2\alpha+5)+7 = 7$

Similarly $\beta^3+4\beta^2-7\beta+35 = \beta\left(\beta^2-2\beta+5\right)+6\beta^2-12\beta+35 = 6\left(\beta^2-2\beta+5\right)+5=5$

$D_1 : D_2$ Equation is $x^2-12x+35=0$

Illustration 17: If $y=ax^2+bx+c>0\ \forall\ x\in R$, then prove that polynomial $z = y+\frac{dy}{dx}+\frac{d^2y}{dx^2}$ will also be greater than 0. **(JEE ADVANCED)**

Sol: In this problem, the given equation $y=ax^2+bx+c>0\ \forall\ x\in R$ means $a>0$ & $b^2-4ac<0$. Hence, by substituting y in $z = y+\frac{dy}{dx}+\frac{d^2y}{dx^2}$ and solving we will get the result.

Since, $y>0 \Rightarrow a>0$ & $b^2-4ac<0$

$Z = ax^2+bx+c+2ax+b+2a = ax^2+(b+2a)x+b+c+2a$

Again, as $a>0$ & $b^2-4ac<0$

$D=(b+2a)^2-4a(b+c+2a) = b^2-4ac-4a^2<0$

For the new expression since D < 0 and a > 0, it is always positive.

Illustration 18: If a Quadratic equation (QE) is formed from $y^2=4ax$ & $y=mx+c$ and has equal roots, then find the relation between c, a & m. **(JEE MAIN)**

Sol: By solving these two equations, we get the quadratic equation; and as it has equal roots, hence D = 0.

$(mx+c)^2=4ax;$ $\qquad m^2x^2+2(cm-2a)x+c^2=0$

Given that the roots are equal. So, D = 0 $\Rightarrow 4(cm-2a)^2 \Rightarrow 4c^2m^2 \Rightarrow 4a^2=4acm$

$a=cm \Rightarrow c=\frac{a}{m};$

This is a condition for the line y = mx + c to be a tangent to the curve $y^2=4ax$.

Illustration 19: Prove that the roots of the equation $ax^2+bx+c=0$ are given by $\frac{2c}{-b\mp\sqrt{b^2-4ac}}$ **(JEE MAIN)**

Sol: We know that the roots of the quadratic equation $ax^2+bx+c=0$ are found by $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Therefore, in multiplying and dividing by $-b\mp\sqrt{b^2-4ac}$ we can prove the above problem.

$ax^2+bx+c=0$

$$\Rightarrow x^2+\frac{b}{a}x+\frac{c}{a}=0 \qquad \Rightarrow \left(x+\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

$$\Rightarrow \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2} \qquad \Rightarrow \left(x-\frac{b}{2a}\right)=\pm\left(\frac{\sqrt{b^2-4ac}}{2a}\right)$$

$$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\times\frac{-b\mp\sqrt{b^2-4ac}}{-b\mp\sqrt{b^2-4ac}}$$

$$\Rightarrow x=\frac{(-b)^2-\left(b^2-4ac\right)}{2a\left(-b\mp\sqrt{b^2-4ac}\right)} \Rightarrow x=\frac{2c}{-b\pm\sqrt{b^2-4ac}}$$

Illustration 20: Let $f(x) = ax^2 + bx + a$ which satisfies the equation $f\left(x+\frac{7}{4}\right) = f\left(\frac{7}{4}-x\right)$ and the equation $f(x) = 7x + a$ has only one solution. Find the value of (a + b). **(JEE ADVANCED)**

Sol: As $f(x) = 7x + a$ has only one solution, i.e. D = 0 and $f\left(x+\frac{7}{4}\right) = f\left(\frac{7}{4}-x\right)$. Hence, by solving these two equations simultaneously we will get the values of a and b.

Given

$f(x) = ax^2 + bx + a$ ...(i)

$f\left(x+\frac{7}{4}\right) = f\left(\frac{7}{4}-x\right)$ ...(ii)

and given that $f(x) = 7x + a$ ...(iii)

has only one solution. Now using (i) and (ii).

$$a\left(x+\frac{7}{4}\right)^2 + b\left(x+\frac{7}{4}\right) + a = a\left(\frac{7}{4}-x\right)^2 + b\left(\frac{7}{4}-x\right) + a \Rightarrow a\left(x^2 + \frac{49}{16} + \frac{7}{2}x\right) + b\left(x+\frac{7}{4}\right) = a\left(\frac{49}{16} + x^2 - \frac{7}{2}x\right) + b\left(\frac{7}{4}-x\right)$$

$\Rightarrow 7ax + 2bx = 0$; $(7a + 2b)x = 0$ ... (iv)

$f(x) = 7x + a$ has only one solution, i.e., D is equals to zero.

$ax^2 + bx + a = 7x + a \Rightarrow ax^2 + (b-7)x = 0 \Rightarrow D = (b-7)^2 - 4a\times 0 \Rightarrow D = x^2(b-7)^2 = 0$; $b = 7$

Using equation (iv), a = –2, Then a + b = 5

Illustration 21: If the equation $2x^2 + 4xy + 7y^2 - 12x - 2y + t = 0$ where t is a parameter that has exactly one real solution of the form (x, y). Find the value of (x + y). **(JEE ADVANCED)**

Sol: As the given equation has exactly one real solution, hence D = 0.

$2x^2 + 4x(y-3) + 7y^2 - 2y + t = 0$

D = 0 (for one solution)

$\Rightarrow 16(y-3)^2 - 8(7y^2 - 2y + t) = 0 \Rightarrow 2(y-3)^2 - (7y^2 - 2y + t) = 0$

$\Rightarrow 2(y^2 - 6y + 9) - (7y^2 - 2y + t) = 0 \Rightarrow -5y^2 - 10y + 18 - t = 0$

$\Rightarrow 5y^2 + 10y + t - 18 = 0$

Again D = 0 (for one solution) $\Rightarrow 100 - 20(t - 18) = 0$

$\Rightarrow 5 - t + 18 = 0$; For $t = 23, 5y^2 + 10y + 5 = 0$

$b^2 - 4ac = 0 \Rightarrow b^2 = 4ac$

For $y = -1$; $2x^2 - 16x + 32 = 0 \quad \therefore x = 4 \Rightarrow x + y = 3$

6. GRAPHICAL APPROACH

Let $y = ax^2 + bx + c$; $y = a\left(\left(x+\frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right)$ a, b and c are real coefficients. ...(i)

Equation (i) represents a parabola with vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ and axis of the parabola is $x = \frac{-b}{2a}$

If a > 0, the parabola opens upward, while if a < 0, the parabola opens downward.

The parabola intersects the x-axis at points corresponding to the roots of $ax^2 + bx + c = 0$. If this equation has

(a) D > 0 the parabola intersects x – axis at two real and distinct points.

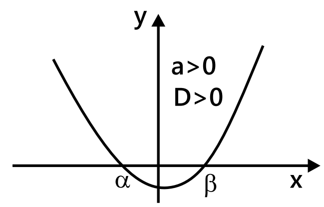
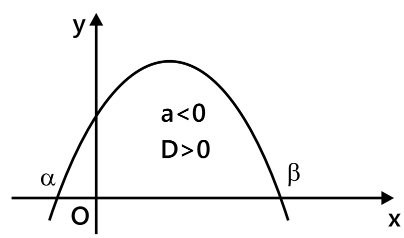
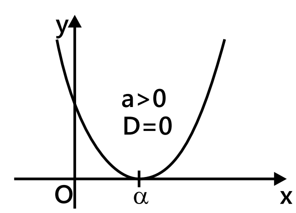
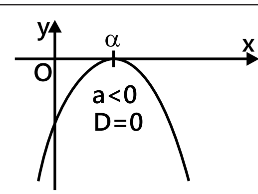
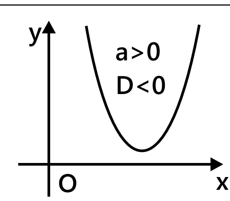
(b) D = 0 the parabola meets x-axis at $x = \frac{-b}{2a}$

(c) D < 0 then;

If a > 0, parabola completely lies above x-axis.

If a < 0 parabola completely lies below x-axis.

Some Important Cases: If $f(x) = ax^2 + bx + c = 0$ and α, β are the roots of f (x)

1.	If a> 0 and D > 0, then $f(x) > 0 \ \forall \in (-\infty, \alpha) \cup (\beta, \infty)$ (where $\alpha < \beta$, and are the roots of $ax^2 + bx + c = 0$)	 **Figure 2.9:** Roots are real & distinct
2.	If a < 0 and D > 0 then $f(x) < 0 \ \forall \ x \in (-\infty, \alpha) \cup (\beta, \infty)$ where $\beta > \alpha$	 **Figure 2.10:** Roots are real & distinct
3.	If a>0 and D = 0 then $\alpha = \beta$ $f(x) > 0 \ \forall \ x : x \neq \alpha$ $f(\alpha) = 0$	 **Figure 2.11:** Roots are real & equal
4.	If a< 0 and D = 0 then $px^2 + qx + r = 0$ and $f(x) < 0 \ \forall \ x \neq \alpha$ $f(\alpha) = 0$	 **Figure 2.12:** Roots are real & equal
5.	If a> 0 and D < 0 then $f(x) > 0 \ \forall \ x \in R$	 **Figure 2.13:** Roots are complex

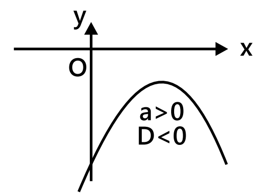
6.	If a< 0 and D < 0 then $f(x) < 0\ \forall\ x \in R$	 **Figure 2.14:** Roots are complex

Illustration 22: The graph of a quadratic polynomial $y = ax^2 + bx + c$ is as shown in the figure below. Comment on the sign of the following quantities. **(JEE MAIN)**

(A) b – c (B) bc (C) c – a (D) ab^2

Sol: Here a < 0;

$-\frac{b}{a} < 0 \Rightarrow b < 0; \frac{c}{a} < 0 \Rightarrow c > 0.$ As b – c = (–ve) – (+ve); it must be negative;

Also, bc = (–ve)(+ve); this must be negative;

Then, $\beta + \frac{1}{\alpha}$ = (–ve) (+ve); the product must be negative; finally,

c – a = (+ve) – (–ve), it must be positive.

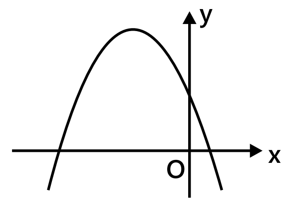


Figure 2.15

Illustration 23: Suppose the graph of a quadratic polynomial $y = x^2 + px + q$ is situated so that it has two arcs lying between the rays y = x and y = 2x, x $\geq$ 0. These two arcs are projected onto the x-axis yielding segments S_L and S_R, with S_R to the right of S_L. Find the difference of the length (S_R)-(S_L) **(JEE MAIN)**

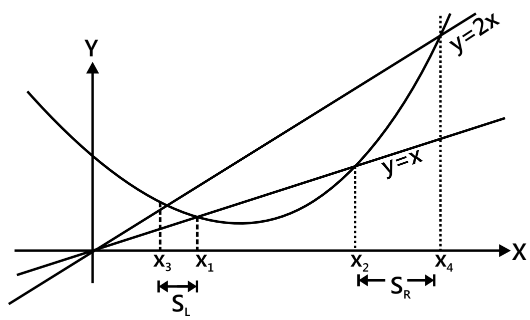


Figure 2.16

Sol: Let the roots of $x^2 + px + q = x$ be x_1 and x_2 and the roots of $x^2 + px + q = 2x$ be x_3 and x_4.

$S_R = x_4 - x_2$ and $S_L = x_1 - x_3 \Rightarrow S_R - S_L = x_4 + x_3 - x_1 - x_2$.

$\therefore l(S_R) - l(S_L) = \left[-(p-2) - \{-(p-1)\}\right] = 1$

7. THEORY OF EQUATIONS

Consider α, β, γ the roots of $ax^3 + bx^2 + cx + d = 0$; then

$$ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma)$$

$$ax^3 + bx^2 + cx + d = a\left(x^2 - (\alpha+\beta)\right)(x+\alpha\beta)(x-\gamma)$$

$$ax^3 + bx^2 + cx + d = a\left(x^3 - x^2(\alpha+\beta+\gamma) + x(\alpha\beta+\beta\gamma+\gamma\alpha) - \alpha\beta\gamma\right)$$

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} \equiv \left(x^3 - x^2(\alpha+\beta+\gamma) + x(\gamma+\beta\gamma+\gamma\alpha) - \alpha\beta\gamma\right)$$

Comparing them, $\alpha+\beta+\gamma = \frac{-b}{a} \Rightarrow -\frac{\text{coefficient of } x^2}{\text{coefficient of } x^3}, \alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a} \Rightarrow \frac{\text{coefficient of } x}{\text{coefficient of } x^3}$

$\alpha\beta\gamma = -\frac{d}{a} \Rightarrow -\frac{\text{constant term x}}{\text{coefficient of } x^2}$

Similarly for $ax^4 + bx^3 + cx^2 + dx + e = 0$;

$\sum\alpha = \frac{-b}{a}$; $\sum\alpha\beta = \frac{c}{a}$; $\sum\alpha\beta\gamma = -\frac{d}{a}$; $\alpha\beta\gamma\delta = \frac{e}{a}$

MASTERJEE CONCEPTS

As a general rule

$a_0X^n + a_1X^{n-1} + a_2X^{n-2} + a_3X^{n-3} + ... + a_n = 0$ has roots $X_1, X_2, X_3 ......X_n$

$\Sigma X_1 = \frac{-a_1}{a_0} = -\frac{\text{coefficient of } X^{n-1}}{\text{coefficient of } X^n}$, $\sum X_1X_2 = \frac{a_2}{a_0} = \frac{\text{coefficient of } X^{n-2}}{\text{coefficient of } X^n}$

$\sum X_1X_2X_3 = -\frac{a_3}{a_0} = -\frac{\text{coefficient of } X^{n-3}}{\text{coefficient of } X^n}$, $X_1X_2X_3....x_n = (-1)^n \frac{\text{constant term}}{\text{coefficient of } X^n} = (-1)^n \frac{a_n}{a_0}$

Vaibhav Krishan (JEE 2009 AIR 22)

Illustration 24: For $ax^2 + bx + c = 0$, x_1 & x_2 are the roots. Find the value of $(ax_1 + b)^{-3} + (ax_2 + b)^{-3}$ **(JEE MAIN)**

Sol: As x_1 and x_2 are the roots of equation $ax^2 + bx + c = 0$, hence, $x_1 + x_2 = \frac{-b}{a}$ and $x_1x_2 = \frac{c}{a}$.

Therefore, by substituting this we will get the result $\frac{1}{(ax_1 + b)^3} + \frac{1}{(ax_2 + b)^3}$

Now $\alpha\beta\gamma = \frac{-d}{a}$, $\alpha\beta + \beta\gamma + \lambda\alpha = \frac{c}{a}$

$$\Rightarrow \frac{1}{(ax_1 + b)^3} + \frac{1}{(ax_2 + b)^3} - \frac{1}{-a^3x_2^3} + \frac{1}{-a^3x_1^3} = \frac{x_1^3 + x_2^3}{a^3x_1^3x_2^3} = \frac{(x_1 + x_2)^3 - 3x_1x_2(x_1 + x_2)}{-c^3} = \frac{-3b}{a^2c^2} + \frac{b^3}{a^3c^3} = \frac{b^3 - 3abc}{a^3c^3}$$

Illustration 25: If the two roots of cubic equation $x^3 + px^2 + qx + r = 0$ are equal in magnitude but opposite in sign, find the relation between p, q, and r. **(JEE MAIN)**

Sol: Considering α, - α and β to be the roots and using the formula for the sum and product of roots, we can solve above problem.

Let us assume the roots are α, - α and β

Then, $\alpha - \alpha + \beta = -p \Rightarrow \beta = -p$

$-\alpha^2 - \alpha p + \alpha p = q \Rightarrow \alpha^2 = -q$; $\quad -\alpha^2\beta = -r \quad \Rightarrow pq = r$

Illustration 26: If the roots of a quadratic equation $(a-b)x^2 + (b-c)x + (c-a) = 0$ are equal then prove $2a = (b + c)$ **(JEE MAIN)**

Sol: In this problem, the sum of all the coefficients is 0, hence its roots are 1 and $\frac{c-a}{a-b}$. Therefore, by using the product of roots formula we can prove the above problem.

As x = 1 is a root of the equation (since sum of all coefficients is 0)

$\therefore$ The other root is also 1

$\therefore$ Product = 1 = $\frac{c-a}{a-b}$; $\therefore$ a – b = c – a $\therefore$ 2a = b + c

Illustration 27: If the roots of p (q – r) x^2 + q(r – p)x + r(p – q) = 0 has equal roots, prove that $\frac{2}{q}=\frac{1}{p}+\frac{1}{r}$ **(JEE MAIN)**

Sol: This problem can be solved in the manner shown in the previous illustration.

One root is 1 ; $\therefore$ other root is 1

$\therefore$ Product = 1 = $\frac{rp-rq}{pq-pr}$; $\therefore$ pq – pr = rp – rq

$\therefore$ q(p + r) = 2rp $\therefore \frac{2}{q}=\frac{p+r}{pr}=\frac{1}{r}+\frac{1}{p}$

Illustration 28: If α, β, γ are the roots of cubic $x^3+qx+r=0$ then find the value of $\Sigma(\alpha-\beta)^2$ **(JEE MAIN)**

Sol: As we know, if α, β and γ are the roots of cubic equation $ax^3+bx^2+cx+d=0$

then $\alpha+\beta+\gamma=\frac{-b}{a}$, $\alpha\beta+\beta\gamma+\lambda\alpha=\frac{c}{a}$ and $\alpha\beta\gamma=\frac{-d}{a}$. Therefore, by using these formulae we can solve the above illustration.

$\alpha+\beta+\gamma=0$; $\sum\alpha\beta=q$; $\alpha\beta\gamma=-r$

$(\alpha+\beta+\gamma)^2=0\Rightarrow\alpha^2+\beta^2+\gamma^2=-2(\alpha\beta+\beta\gamma+\gamma\alpha)\Rightarrow\sum\alpha^2=-2\sum\alpha\beta$

Now $\sum(\alpha-\beta)^2=\sum(\alpha^2+\beta^2-2\alpha\beta)=2\left(\sum\alpha^2-\sum\alpha\beta\right)=-6\sum\alpha\beta=-6q$

Illustration 29: Form the cubic equation whose roots are greater by unity than the roots of $x^3-5x^2+6x-3=0$ **(JEE ADVANCED)**

Sol: By using $x^3-x^2(\Sigma\alpha)+x(\Sigma\alpha_1\beta_1)-\alpha_1\ \beta_1\ \gamma_1=0$ we can form cubic equation.

Here $\alpha_1=\alpha+1$ $\beta_1=\beta+1$ $\gamma_1=\gamma+1$ and α, β and γ are the roots of $x^3-5x^2+6x-3=0$.

$\alpha+\beta+\gamma=5$; $\Sigma\ \alpha\beta=6$; $\alpha\beta\gamma=3$

Let the roots of the new equation be $\alpha_1,\beta_1,\gamma_1$

$\therefore$ The equation is $x^3-x^2\ (\Sigma\alpha)+x(\Sigma\alpha_1\beta_1)-\alpha_1\ \beta_1\ \gamma_1=0$

$\alpha_1=\alpha+1$ $\beta_1=\beta+1$ $\gamma_1=\gamma+1$

$\Sigma\ \alpha_1=\alpha+\beta+\gamma+3=8$

$\Sigma\ \alpha_1\beta_1=\alpha_1\beta_1+\alpha_1\gamma_1+\beta_1\gamma_1=(\alpha+1)(\beta+1)+(\beta+1)(\gamma+1)+(\gamma+1)(\alpha+1)=19$

$\alpha_1\beta_1\gamma_1=(\alpha+1)(\beta+1)(\gamma+1)=15$

$\therefore\ x^3-8x^2+19x-15=0$

Alternate Method

$y = x + 1 \Rightarrow x = y - 1$ Put $(y - 1)$ in given equation

$\Rightarrow (y-1)^3 - 5(y-1)^2 + 6(y-1) - 3 = 0 \Rightarrow y^3 - 1 - 3y^2 + 3y - 5y^2 - 5 + 10y + 6y - 6 - 3 = 0 \Rightarrow y^3 - 8y^2 + 19y - 15 = 0$

Illustration 30: Find the sum of the squares and the sum of the cubes of the roots of $x^3 - ax^2 + bx - c = 0$ **(JEE ADVANCED)**

Sol: Similar to the previous problem.

$\Sigma\alpha = a$; $\Sigma\,\alpha\beta = b$; $\Sigma\,\alpha\beta\gamma = c$;

$\therefore \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2\,\Sigma\,\alpha\beta = a^2 - 2b$

$\alpha^3 + \beta^3 + \gamma^3 = (\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma) + 3\alpha\beta\gamma$

$= (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) + 3\alpha\beta\gamma = (a)(a^2 - 2b - b) + 3c = (a)(a^2 - 3b) + 3c$

Illustration 31: If α, β, γ & δ are the roots of equation $\tan\left(\frac{\pi}{4} + x\right) = 3\tan 3x$, then find the value of $\Sigma \tan\alpha$ **(JEE ADVANCED)**

Sol: Here, $\frac{1 + \tan x}{1 - \tan x} = 3\left(\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}\right)$, therefore by putting tan x = y and solving we will get the result.

The given equation is: $\frac{1 + \tan x}{1 - \tan x} = 3\left(\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}\right)$; Let $\tan x = y \Rightarrow \frac{1 + y}{1 - y} = \frac{3(3y - y^3)}{1 - 3y^2}$

$\Rightarrow 1 - 3y^2 + y - 3y^3 = 9y - 3y^3 - 9y^2 + 3y^4 \ (y \neq 1)$

$\Rightarrow 3y^4 - 6y^2 + 8y - 1 = 0$; $\Sigma y_1 = 0 \Rightarrow \Sigma \tan\alpha = 0$

Illustration 32: Find the number of quadratic equations with real roots remain unchanged even after squaring their roots. **(JEE ADVANCED)**

Sol: As given $\alpha\beta = \alpha^2\beta^2$ and $\alpha^2 + \beta^2 = \alpha + \beta$, therefore by solving it we will get the values of α and β.

$\alpha\beta = \alpha^2\beta^2$ .....(i)

and $\alpha^2 + \beta^2 = \alpha + \beta$ .....(ii)

Hence, $\alpha\beta(1 - \alpha\beta) = 0 \Rightarrow \alpha = 0$ or $\beta = 0$ or $\alpha\beta = 1$

If $\alpha = 0$ then from (ii), $\beta = 0$ or $\beta = 1 \Rightarrow$ roots are (0, 0) or (0,1)

If $\beta = 0$ then, $\alpha = 0$ or $\alpha = 1 \Rightarrow$ roots are (0,0) or (1,0)

If $\beta = \frac{1}{\alpha}$ then $\alpha^2 + \frac{1}{\alpha^2} = \alpha + \frac{1}{\alpha} \Rightarrow \left(\alpha + \frac{1}{\alpha}\right)^2 - 2 = \alpha + \frac{1}{\alpha}$

Hence $t^2 - t - 2 = 0 \Rightarrow (t - 2)(t + 1) = 0 \Rightarrow t = 2$ or $t = -1$

If t = 2 $\Rightarrow \alpha = 1$ and $\beta = 1$, if t = –1 roots are imaginary ω or ω^2

$\therefore$ The number of quadratic equations is one.

MASTERJEE CONCEPTS

The relation between Roots and Coefficients.

If the roots of a quadratic equation $ax^2+bx+c=0\,(a\neq 0)$ are α and β then:

- $\left(\alpha-\beta\right)=\pm\sqrt{\left(\alpha+\beta\right)^2-4\alpha\beta}=\pm\dfrac{\sqrt{b^2-4ac}}{a}=\dfrac{\pm\sqrt{D}}{a}$
- $\alpha^2+\beta^2=\left(\alpha+\beta\right)^2-2\alpha\beta=\dfrac{b^2-2ac}{a^2}$
- $\alpha^2-\beta^2=\left(\alpha+\beta\right)\left(\alpha-\beta\right)=\pm\dfrac{b\sqrt{b^2-4ac}}{a^2}$
- $\alpha^3+\beta^3=\left(\alpha+\beta\right)^2-3\alpha\beta\left(\alpha+\beta\right)-\dfrac{b\left(b^2-3ac\right)}{a^3}$
- $\alpha^3-\beta^3=\left(\alpha-\beta\right)^3+3\alpha\beta\left(\alpha-\beta\right)=\left(\alpha-\beta\right)\left[\left(\alpha+\beta\right)^2-4\alpha\beta+3\alpha\beta\right]=\pm\dfrac{(b^2-ac)\sqrt{b^2-4ac}}{a^3}$
- $\alpha^4+\beta^4=\left(\alpha^2+\beta^2\right)-2\alpha^2\beta^2=\left(\dfrac{b^2-2ac}{a^2}\right)-2\dfrac{c^2}{a^2}$
- $\alpha^4-\beta^4=(\alpha^2-\beta^2)(\alpha^2+\beta^2)=\dfrac{\pm b(b^2-2ac)\sqrt{b^2-4ac}}{a^4}$
- $\alpha^2+\alpha\beta+\beta^2=\left(\alpha+\beta\right)^2-\alpha\beta=\left(b^2-ac\right)/a^2$
- $\alpha^2\beta+\beta^2\alpha=\alpha\beta\left(\alpha+\beta\right)=-bc/a^2$
- $\left(\dfrac{\alpha}{\beta}\right)^2+\left(\dfrac{\beta}{\alpha}\right)^2=\dfrac{\alpha^4+\beta^4}{\alpha^2\beta^2}=\dfrac{(\alpha^2+\beta^2)^2-2\alpha^2\beta^2}{\alpha^2\beta^2}=(b^2-2ac/ac)^2-2$

Nitish Jhawar (JEE 2009 AIR 7)

8. TRANSFORMATION OF EQUATIONS

We now list some of the rules to form an equation whose roots are given in terms of the roots of another equation.

Let the given equation be $a_0x^n+a_1x^{n-1}+....a_{n-1}x+a_n=0$ .... (i)

Rule 1: To form an equation whose roots are k(≠0) times the roots of the equation, replace x by $\dfrac{x}{k}$.

Rule 2: To form an equation whose roots are the negatives of the roots in the equation, replace x by –x.

In rule 1, y = kx Hence x = y/k. Now replace x by y/k and form the equation. We can do the same thing for the other rules.

Alternatively, change the sign of the coefficients of $X^{n-1}, X^{n-3}, X^{n-5}$,.... etc. in (i).

Rule 3: To form an equation whose roots are k more than the roots of the equation, replace x by x – k.

Rule 4: To form an equation whose roots are reciprocals of the roots of the equation, replace x by

$\dfrac{1}{x}\ (x\neq 0)$ and then multiply both sides by x^n .

Rule 5: To form an equation whose roots are the square of the roots of the equation in (1) proceed as follows:

Step 1 Replace x by $\sqrt{x}$ in (1)

Step 2 Collect all the terms involving $\sqrt{x}$ on one side.

Step 3 Square both the sides and simplify.

For instance, to form an equation whose roots are the squares of the roots of $\frac{(\alpha+\beta)(\alpha\beta)\pm\alpha\beta\sqrt{(\alpha+\beta)^2-4(\alpha\beta)^3}}{2}$ replace x by $\sqrt{x}$ to obtain.

$$x\sqrt{x}+2x-\sqrt{x}+2=0 \Rightarrow \sqrt{x}\,(x-1)=-2(x+1)$$

Squaring both sides, we get $x(x-1)^2=4(x+1)^2$ or $x^3-6x^2-7x-4=0$

Rule 6: To form an equation whose roots are the cubes of the roots of the equation, proceed as follows:

Step 1 Replace x by $x^{1/3}$

Step 2 Collect all the terms involving $x^{1/3}$ and $x^{2/3}$ on one side.

Step 3 Cube both the sides and simplify.

9. CONDITION FOR MORE THAN 2 ROOTS

To find the condition that a quadratic equation has more than 2 roots.

$ax^2+bx+c=0$ Let α,β,γ be the roots of the equation

$a\alpha^2+b\alpha+c=0$ ... (i)

$a\beta^2+b\beta+c=0$ ... (ii)

$a\gamma^2+b\gamma+c=0$ ... (iii)

Subtract (ii) from (i) $a(\alpha^2-\beta^2)+b(\alpha-\beta)=0$; $(\alpha-\beta)(a(\alpha+\beta)+b)=0$

$\Rightarrow a(\alpha+\beta)+b=0 \because \alpha\neq\beta$ ...(iv)

Subtract (iii) from (ii) $\Rightarrow a(\beta+\gamma)+b=0$ ...(v)

Subtract (i) from (iii) $\Rightarrow a(\gamma+\alpha)+b=0$

Subtract (v) from (iv) $\Rightarrow a(\gamma+\beta-\beta-\alpha)=0$ or $a(\gamma-\alpha)=0 \Rightarrow a=0$

Keeping $a=0$ in (iv) ; b = 0 and c = 0 $\Rightarrow$ It is an identity

Illustration 33: If $(a^2-1)x^2+(a-1)x+a^2-4a+3=0$ is an identity in x, then find the value of a. **(JEE MAIN)**

Sol: The given relation is satisfied for all real values of x, so all the coefficients must be zero.

$$\left.\begin{aligned} a^2-1=0&\Rightarrow a=\pm1\\ a-1=0&\Rightarrow a=1\\ a^2-4a+3=0&\Rightarrow 1,3\end{aligned}\right\}\text{Common value a is 1}$$

Illustration 34: If the equation $a(x-1)^2+b(x^2-3x+2)+x-a^2=0$ is satisfied for all x $\in$ R, find all possible ordered pairs (a, b). **(JEE ADVANCED)**

Sol: Similar to illustration 33, we can solve this illustration by taking all coefficients to be equal to zero.

$$a(x-1)^2+b(x^2-3x+2)+x-a^2=0$$

$\Rightarrow (a+b)x^2 - (2a+3b-1)x + 2b - a^2 + a = 0$

Since the equation is satisfied for all α, it becomes an identity

Coeff. of x^2 = 0	**Coeff. of x = 0**	**Constant term = 0**
a + b = 0	2a + 3b – 1 = 0	$2b - a^2 + a = 0$; $2 - a^2 + a = 0$
a = –b .....(i)	using (i) ; $\Rightarrow$ –2b + 3b = 1 ;	$a^2 - a - 2 = 0$
	$\Rightarrow$ b = 1	$\Rightarrow$ (a + 1) (a – 2) = 0 ; a = –1, 2

But from (i) a = –b $\Rightarrow$ only a = –1 is the possible solution. Hence (a, b) = (–1, 1)

10. SOLVING INEQUALITIES

10.1 Intervals

Given E(x) = (x – a)(x – b)(x – c)(x – d) ≥ 0

To find the solution set of the above inequality we have to check the intervals in which E(x) is greater/less than zero.

Intervals

Closed interval **Open interval** **Open-closed interval** **Closed-open interval**

(a) **Closed Interval:** The set of all values of x, which lies between a & b and is also equal to a & b is known as a closed interval, i.e. if $a \leq x \leq b$ then it is denoted by $x \in [a, b]$.

(b) **Open Interval:** The set of all values of x, which lies between a & b but equal to a & b is known as an open interval, i.e. if $a < x < b$ then it is denoted by $a \in (a, b)$

(c) **Open-Closed Interval:** The set of all values of x, which lies between a & b, equal to b, but not equal to a is known as an open-closed interval, i.e. if $a < x \leq b$ then it is denoted by $x \in (a, b]$.

(d) **Closed-open Interval:** The set of all values of x, which lies between a & b, equal to a but not equal to b is called a closed-open interval, i.e. if $a \leq x < b$, then it is denoted by $x \in [a, b)$.

Note: (i) $x \geq a \Rightarrow [a, \infty)$ (ii) $x > a \Rightarrow (a, \infty)$

(iii) $x \leq a \Rightarrow (-\infty, a)$ (iv) $x < a \Rightarrow (-\infty, a)$

10.2 Some Basic Properties of Intervals

(a) In an inequality, any number can be added or subtracted from both sides of inequality.

(b) Terms can be shifted from one side to the other side of the inequality. The sign of inequality does not change.

(c) If we multiply both sides of the inequality by a non-zero positive number, then the sign of inequality does not change. But if we multiply both sides of the inequality by a non-zero negative number then the sign of the inequality does get changed.

(d) In the inequality, if the sign of an expression is not known then it cannot be cross multiplied. Similarly, without knowing the sign of an expression, division is not possible.

(i) $\frac{x-2}{x-5} > 1 \Rightarrow x-2 > x-5$ (Not valid because we don't know the sign of the expression)

(ii) $\frac{x-2}{(x-5)^2} > 1 \Rightarrow (x-2) > (x-5)^2$ (valid because $(x-5)^2$ is always positive)

10.3 Solution of the Inequality

(a) Write all the terms present in the inequality as their linear factors in standard form i.e. $x \pm a$.

(b) If the inequality contains quadratic expressions, $f(x) = ax^2 + bx + c$; then first check the discriminant $(D = b^2 - 4ac)$

(i) If D > 0, then the expression can be written as f(x) = a (x - α)(x - β). Where α and β are given by $\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(ii) If D = 0, then the expression can be written as $f(x) = a(x - \alpha)^2$, where $\alpha = \frac{-b}{2a}$.

(iii) If D < 0 & if

- a > 0, then f(x) > 0 $\forall X \in R$ and the expression will be cross multiplied and the sign of the inequality will not change.
- a < 0, then f(x) < 0 $\forall X \in R$ and the expression will be cross multiplied and the sign of the inequality will change.
- If the expression (say 'f') is cancelled from the same side of the inequality, then cancel it and write $f \neq 0$ e.g.,

(i) $\frac{(x-2)(x-3)}{(x-2)(x-5)} > 1 \Rightarrow \frac{(x-3)}{(x-5)} > 1$ iff $x - 2 \neq 0$

(ii) $\frac{(x-5)^2(x-8)}{(x-5)} \geq 0 \Rightarrow (x-5)(x-8) \geq 0$ iff $x - 5 \neq 0$

(iii) Let $f(x) = \frac{(x-a_1)^{k_1}(x-a_2)^{k_2}....(x-a)^{k_n}}{(x-b_1)^{r_1}(x-b_2)^{r_2}....(x-b)^{r_n}}$

Where k_1, k_2 .... k_n & r_1, r_2 ....r_n $\in$ N and a_1, a_2, ...a_n & b_1, b_2 ....b_n are fixed real numbers. The points where the numerator becomes zero are called zeros or roots of the function and points where the denominator becomes zero are called poles of the function. Find poles and zeros of the function f(x). The corresponding zeros are a_1, a_2, ...a_n and poles are b_1, b_2 ....b_n. Mark the poles and zeros on the real numbers line. If there are n poles & n zeros the entire number line is divided into 'n+1' intervals. For f(x), a number line is divided into '2n+1' intervals.

Place a positive sign in the right-most interval and then alternate the sign in the neighboring interval if the pole or zero dividing the two interval has appeared an odd number of times. If the pole or zero dividing the interval has appeared an even number of times then retain the sign in the neighboring interval. The solution of f(x) > 0 is the union of all the intervals in which the plus sign is placed, and the solution of f(x) < 0 is the union of all the intervals in which minus sign is placed. This method is known as the WAVY CURVE method.

Now we shall discuss the various types of inequalities.

Type I: Inequalities involving non-repeating linear factors (x – 1) (x – 2) ≥ 0

1st condition $\left.\begin{array}{l}(x-1) > 0 \Rightarrow x > 1 \\ (x-2) > 0 \Rightarrow x > 2\end{array}\right\} x \geq 2$

2nd condition $\left.\begin{array}{l}x-1 < 0 \Rightarrow x < 1 \\ x-2 < 0 \Rightarrow x < 2\end{array}\right\} x \leq 1$

$\therefore x \in (-\infty, 1] \cup [2, \infty)$

Illustration 35: (x – 3) (x+1) $\left(x-\frac{12}{7}\right)$ < 0, find range of x **(JEE MAIN)**

Sol: Comparing all brackets separately with 0, we can find the range of values for x.

$$x<-1 \text{ and } \frac{12}{7}<x<3;\ \therefore x\in(-\infty,-1)\cup\left(\frac{12}{7},3\right)$$

Type II: Inequalities involving repeating linear factors

$$(x-1)^2(x+2)^3(x-3)\le 0$$

$$\Rightarrow\ (x+1)^2(x+2)^2(x+2)(x-3)\le 0$$

$\Rightarrow$ (x+2)(x – 3) $\leq$ 0 ; x $\in$ [-2, 3]

Illustration 36: Find the greatest integer satisfying the equation.

$$(x+1)^{101}(x-3)^2(x-5)^{11}(x-4)^{200}(x-2)^{555}<0$$ **(JEE MAIN)**

Sol: Comparing all brackets separately with 0, we can find the greatest integer.

The inequality $\{-(x-2)\}^2-(x-2)-2=0$

$\Rightarrow$ (x + 1)(x – 5)(x – 2) < 0

x = -1, 3 , 5, 4, 2

x $\in$ $(-\infty,-1)$ U (2, 3) U (3, 4) U (4, 5) .

Type III: Inequalities expressed in rational form.

Illustration 37: $\frac{(x-1)(x+2)}{(x+3)(x-4)}\ge 0$ **(JEE MAIN)**

Sol: If $\frac{(x+a)(x+b)}{(x+c)(x+d)}\ge 0$ then $(x+c)(x+d)\ne 0$, and $(x+a)(x+b)=0$

Hence, $x\ne -3,4$ & x = 1, -2 ; $x\in(-\infty,-3)$ U [-2, 1] U (4, ∞)

Illustration 38: $\frac{x^2(x+1)}{(x-3)^3}<0$ **(JEE MAIN)**

Sol: Similar to the illustration above.

$\frac{x+1}{x-3}<0$ x ≠ 3, -1, 0 ; x $\in$ (–1, 0) U (0, 3)

Illustration 39: $\frac{x^2-1}{x^2-7x+12}\ge 1$ **(JEE MAIN)**

Sol: First reduce the given inequalities in rational form and then solve it in the manner similar to the illustration above.

$$\frac{x^2-1}{(x-4)(x-3)}\ge 1$$

$\therefore \dfrac{(x+1)(x-1)}{(x-4)(x-3)} \geq 1 \quad \Rightarrow \dfrac{x^2-1}{x^2-7x+12} - 1 \geq 0$

$\therefore \quad \dfrac{x^2-1-x^2+7x-12}{(x-4)(x-3)} \geq 0 \qquad \therefore \quad \dfrac{7x-13}{(x-4)(x-3)} \geq 0$

$\therefore \quad x \neq 3, 4\ ; \qquad x \in \left[\dfrac{13}{7}, 3\right) \cup (4, \infty)$

Type IV: Double inequality

Illustration 40: $1 < \dfrac{3x^2-7x+8}{x^2+1} \leq 2$ **(JEE ADVANCED)**

Sol: Here $3x^2-7x+8 > x^2+1$ therefore if D < 0 & if a > 0, then f(x) > 0 and always positive for all real x.

$3x^2-7x+8 > x^2+1 \Rightarrow 2x^2-7x+7 > 0\,;$

D= b^2-4ac = 49 – 56 = -7

$\therefore D < 0$ & $a > 0$ $\quad \therefore$ always positive for all real x

$3x^2-7x+8 \leq 2x^2+2 \Rightarrow x^2-7x+6 \leq 0 \Rightarrow (x-1)(x-6) \leq 0$

$x \in [1, 6]\ ;\ x \in [1, 6] \cap R$

Type V: Inequalities involving biquadrate expressions

Illustration 41: $(x^2+3x+1)(x^2+3x-3) \geq 5$ **(JEE ADVANCED)**

Sol: Using $x^2+3x = y$,we can solve this problem

Let $x^2+3x = y \qquad \therefore (y+1)(y-3) \geq 5$

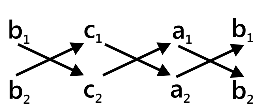
$y^2-2y-8 \geq 0 \qquad \therefore\ (y-4)(y+2) \geq 0$

$\therefore\ (x+4)(x-1)(x+2)(x+1) \geq 0 \Rightarrow x \in (-\infty, -4] \cup [-2, -1] \cup [1, \infty)$

11. CONDITION FOR COMMON ROOTS

Consider that two quadratic equations are $a_1x^2+b_1x+c_1 = 0$ and $a_2x^2+b_2x+c_2 = 0$

(i) One root is common



Let α, be the common root. then α satisfies

$a_1\alpha^2+b_1\alpha+c_1 = 0$

$a_3\alpha^3+b_2\alpha+c_2 = 0$

By cross multiplication method, $\dfrac{\alpha^2}{b_1c_2-b_2c_1} = \dfrac{\alpha}{-(a_1c_2-c_1a_2)} = \dfrac{1}{a_1b_2-b_1a_2}$

$$\frac{\alpha^2}{b_1c_2-b_2c_1} = \frac{\alpha}{c_1a_2-a_1c_2} = \frac{1}{a_1b_2-b_1a_2}$$

$$\alpha^2 = \frac{b_1c_2-b_2c_1}{a_1b_2-b_1a_2} \qquad \text{... (i)}$$

$\alpha = \dfrac{c_1a_2 - a_1c_2}{a_1b_2 - b_1a_2}$ ... (ii)

Divide (1)/(2)

$\alpha = \dfrac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$ ... (iii)

equating (i) and (ii) ; $(c_1 a_2 - c_2 a_1)^2 = (a_1 b_2 - b_1 a_2)(b_1 c_2 - b_2 c_1)$ is the condition for a common root.

(ii) If both roots are common, then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

Illustration 42: Determine the values of m for which the equations $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$ may have a common root. **(JEE MAIN)**

Sol: Consider α to be the common root of the given equations. Then, α must satisfy both the equations. Therefore, by using a multiplication method we can solve this problem.

$3\alpha^2 + 4m\alpha + 2 = 0$; $\quad 2\alpha^2 + 3\alpha - 2 = 0$

Using the cross multiplication method, we have

$(-6 - 4)^2 = (9 - 8m)(-8m - 6)$

$\Rightarrow 50 = (8m - 9)(4m + 3) \Rightarrow 32m^2 - 12m - 77 = 0$

$\Rightarrow 32m^2 - 56m + 44 = 0 \Rightarrow 8m(4m - 7) + 11(4m - 7) = 0$

$\Rightarrow (8m + 11)(4m - 7) = 0 \quad \Rightarrow \quad m = -\dfrac{11}{8}, \dfrac{7}{4}$

Illustration 43: The equation $ax^2 + bx + c$ and $y \geq 0$ have two roots common, Find the value of (a + b). **(JEE ADVANCED)**

Sol: We can reduce $x^3 - 2x^2 + 2x - 1 = 0$ to $(x - 1)(x^2 - x + 1) = 0$ as the given equations have two common roots, therefore $-\omega$ and $-\omega^2$ are the common roots (as both roots of a quadratic equation are either real or non-real).

We have $x^3 - 2x^2 + 2x - 1 = 0 \Rightarrow (x - 1)(x^2 - x + 1) = 0$

$\Rightarrow x = 1$ or $x = -\omega, -\omega^2$, where $\omega = \dfrac{-1 + \sqrt{3}i}{2}$

Since $ax^2 + bx + a = 0$ and $x^3 - 2x^2 + 2x - 1 = 0$ have two roots in common, therefore $-\omega$ and $-\omega^2$ are the common roots (as both roots of a quadratic equation are either real or non-real), also $-\omega$ is a root of $ax^2 + bx + a = 0$. Hence.

$a(1 + \omega^2) - b\omega = 0 \quad \Rightarrow a(-\omega) - b\,\omega = 0$ (as $1 + \omega + \omega^2 = 0$)

$\Rightarrow a + b = 0$

12. MAXIMUM AND MINIMUM VALUE OF A QUADRATIC EQUATION

$y = ax^2 + bx + c$ attains its minimum or maximum value at $x = \dfrac{-b}{2a}$ according to a > 0 or a < 0

MAXIMUM value case

When a < 0 then $y_{max} = \dfrac{-D}{4a}$ i.e. $y \in \left(-\infty, \dfrac{-D}{4a}\right]$

MINIMUM value case

When a > 0 then $y_{min} = \frac{-D}{4a}$ i.e. $y \in \left[\frac{-D}{4a}, \infty\right)$

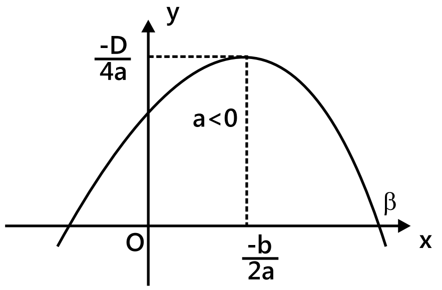


Figure 2.17

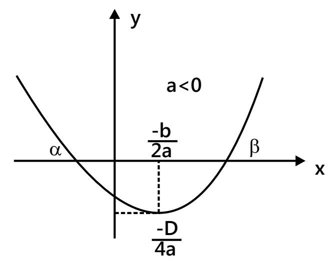


Figure 2.18

MASTERJEE CONCEPTS

If α is a repeated root, i.e., the two roots are α , α of the equation f(x) = 0, then α will be a root of the derived equation f′(x) = 0 where $f'(x) = \frac{df}{dx}$

If α is a repeated root common in f(x) = 0 and ϕ (x) = 0, then α is a common root both in f′(x) = 0 and ϕ ′(x) = 0.

Shrikant Nagori (JEE 2009 AIR 30)

Illustration 44: Find the range of $y = \frac{x}{x^2 - 5x + 9}$ **(JEE ADVANCED)**

Sol: Here as $x \in R$ therefore $D \geq 0$. Hence, by solving these inequalities we can find the required range.

$x = yx^2 - 5yx + 9y$; $\quad yx^2 - 5yx + 9y - x = 0$.

$yx^2 - (5y+1)x + 9y = 0$; $\quad \because x \in R \quad D \geq 0$

$\therefore (5y+1)^2 - 36\,y^2 \geq 0$; $\quad \therefore 25\,y^2 + 1 + 10y - 36\,y^2 \geq 0$

$\therefore -11\,y^2 + 10y + 1 \geq 0 \quad 11y^2 - 11y + y - 1 \leq 0$

$(11y+1)(y-1) \leq 0$; $\quad y \in \left[\frac{-1}{11}, 1\right]$

Putting the end points in the eq. $1 = \frac{x}{x^2 - 5x + 9}$; $\quad x^2 - 6x + 9 = 0 \quad \therefore (x-3)^2 = 0$

If D < 0, then 1 would be open, i.e. excluded; $\frac{-1}{11} = \frac{x}{x^2 - 5x + 9}$

$\Rightarrow -(x^2 - 5x + 9) = 11x$; $\therefore x^2 + 6x + 9 = 0$; $\therefore (x+3)^2 = 0$; $\therefore \frac{-1}{11}$ remains closed

Alternative Solution: $y = \frac{x}{x^2 - 5x + 9} = \frac{1}{\left(x - 5 + \frac{9}{x}\right)}$

Apply the concept of Arithmetic mean ≥ Geometric mean for the values for x and 9/x

We have $\frac{\left(x+\frac{9}{x}\right)}{2} \geq \sqrt{x * \frac{9}{x}}$

Thus $x+\frac{9}{x} \geq 6$ for x ≥ 0 and $x+\frac{9}{x} \leq 6$ for x < 0

Since the term is in the denominator if we consider its maximum value, we will get the minimum value of y and vice versa.

The maximum value of y will be $\frac{1}{6-5}=1$ and

The minimum value of will be $\frac{1}{-6-5} = \frac{-1}{11}$.

Thus the range of y is $\left[\frac{-1}{11}, 1\right]$

Illustration 45: Find range of $y=\frac{x^2+2x-3}{x^2+2x-8}$ **(JEE ADVANCED)**

Sol: Similar to the preceding problem, by taking $b^2 - 4ac \geq 0$ we can solve it.

$x^2 + 2x - 3 = yx^2 + 2xy - 8y$

$(y-1)x^2 + (2y-2)x - (8y-3) = 0$; $b^2 - 4ac \geq 0$

$\therefore (2y-2)^2 + (4)(8y-3)(y-1) \geq 0 \Rightarrow 4y^2 + 4 - 8y + 4(8y^2 - 3y - 8y + 3) \geq 0$

$4y^2 + 4 - 8y - 44y - 32y^2 + 12 \geq 0 \Rightarrow 36y^2 - 52y + 16 \geq 0$

$\therefore 9y^2 - 13y + 4 \geq 0 \Rightarrow (y-1)(9y-4) \geq 0$; $\therefore y \in \left(-\infty, \frac{4}{9}\right] \cup (1, \infty)$

To verify if the bracket is open or closed, apply the end points in the equation,

Check for $y=\frac{4}{9}$; $\frac{4}{9}=\frac{x^2+2x-3}{x^2+2x-8}$

$4x^2+8x-32=9x^2+18x-27$; $\therefore 5x^2+10x+5=0$

$\therefore x^2+2x+1=0$; $\therefore (x+1)^2=0$

$\therefore x=-1$ $\therefore \frac{4}{9}$ is closed

Check for 1, $1=\frac{x^2+2x-3}{x^2+2x-8}$

$\therefore x^2+2x-3=x^2+2x-8$ Since no value of x can be found, 1 is open

Illustration 46: Find the limits of 'a' such that $y=\frac{ax^2-7x+5}{5x^2-7x+a}$ is capable of all the values of 'x' being a real quantity. **(JEE ADVANCED)**

Sol: Similar to Illustration 45.

$5yx^2-7xy+ay=ax^2-7x+5$

$(5y-a)x^2-7x(y-1)+ay-5=0$;

$D = 49\left(y^2+1-2y\right)-4(ay-5)(5y-a)$; $D \geq 0$

$49y2 - 49 - 98y - 20ay^2 + 100y + 4a^2y - 20\,a \geq 0$

$y^2\,(49 - 20a) + y(2 + 4a^2) + 49 - 20\,a \geq 0$

$(2 + 4a^2)^2 - [2(49 - 20a)]^2 \leq 0\,;$ $\quad \therefore\ (2 + 4a^2 + 2(49 - 20a))(2 + 4a^2 - a^2(49 - 20a))$

$(1 + 2a^2 + 49 - 20a)(1 + 2a^2 - 49 + 20\,a) \leq 0$

$(a^2 - 10a + 25)\,(a^2 + 10\,a - 24) \leq 0\,;$ $\quad \therefore\ (a + 12)(a - 2) \leq 0$

13. LOCATION OF ROOTS

Let $f(x) = ax^2 + bx + c;\ \ a, b, c \in \mathbb{R}$ a is not equal to 0 and $\alpha,\ \beta$ be the roots of $f(x) = 0$

Type I: If both the roots of a quadratic equation $f(x) = 0$ are greater than a specified number, say 'd', then

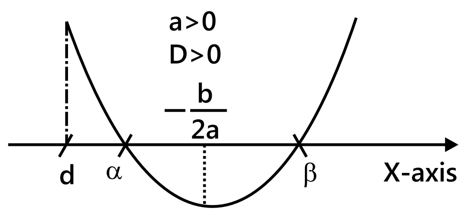


Figure 2.19

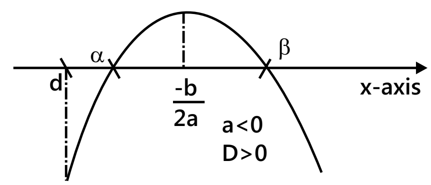


Figure 2.20

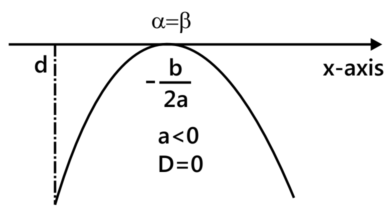


Figure 2.21

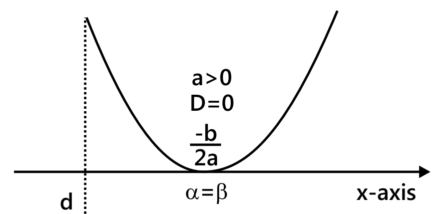


Figure 2.22

(i) $D \geq 0$ (ii) $f(x) > 0$ (iii) $\frac{-b}{2a} > d$

Type II: If both the roots are less than a specified number, say 'd', then

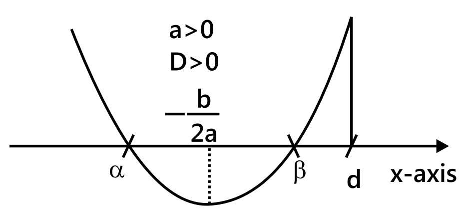


Figure 2.23

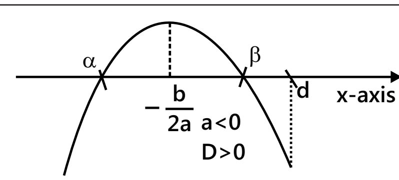


Figure 2.24

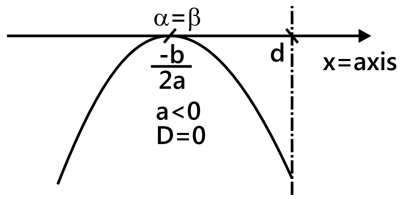


Figure 2.25

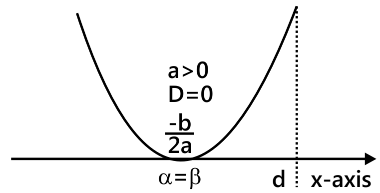


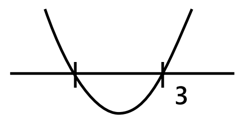
Figure 2.26

(i) $D \geq 0$ (ii) $f(x) > 0$ (iii) $\frac{-b}{2a} < d$

Illustration 47: If both the roots of the quadratic equation $x^2 + x(4-2k) + k^2 - 3k - 1 = 0$ are less than 3, then find the range of values of k. **(JEE MAIN)**

Sol: Here both the roots of the given equation is less than 3, hence, $D \geq 0, \frac{-b}{2a} < 3$ and f (3) > 0.

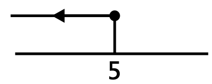
The equation is $f(x) = x^2 + x(4-2k) + k^2 - 3k - 1 = 0$



$D \geq 0$ ...(i)

$\frac{-b}{2a} < 3$ ....(ii)

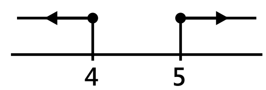
$f(3) > 0$ ....(iii)



(i) $D \geq 0 \Rightarrow (4 - 2k)^2 - 4(k^2 - 3k - 1) \geq 0$

$\Rightarrow$ (k2 – 4k + 4) $-(k^2 - 3k - 1) \geq 0$

$\Rightarrow -k + 5 \geq 0$ $\Rightarrow k - 5 \leq 0$;



$k \in (-\infty, 5]$

(ii) $\frac{-4(4-2k)}{2} < 3$; $k - 2 < 3$; $k < 5$

(iii) $f(3) > 0$ $\Rightarrow 9 + 3(4 - 2k) + k^2 - 3k - 1 > 0$

$\Rightarrow k^2 - 9k + 20 > 0$ $\Rightarrow (k - 4)(k - 5) > 0$

$k \in (-\infty, 4) \cup (5, \infty)$; Combining all values we get $k \in (-\infty, 4)$

Type III: A real number d lies between the roots of $f(x) = 0$ or both the roots lie on either side of a fixed number say 'd' then af(d)<0, and D>0.

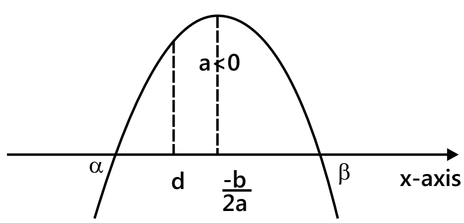


Figure 2.27

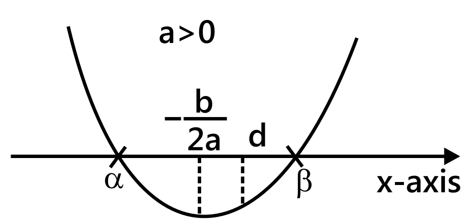


Figure 2.28

Type IV: Exactly one root lies in the interval (d, e) when d < e, then $f(d)\cdot f(e)<0$

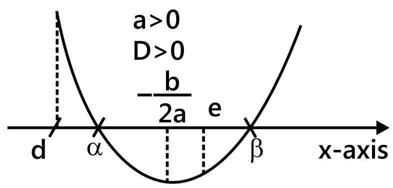


Figure 2.29

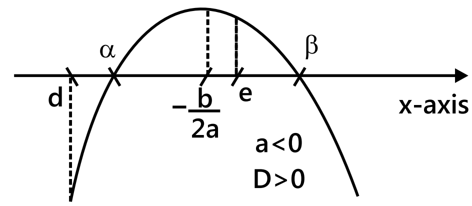


Figure 2.30

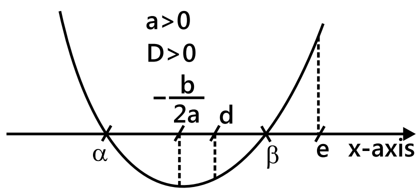


Figure 2.31

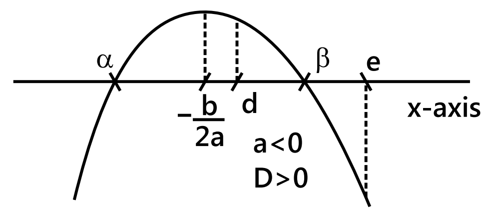


Figure 2.32

Type V: If both the roots of $f(x) = 0$ are confined between real numbers'd' and 'e', where d < e. Then

(i) $D \geq 0$, (ii) $f(d)f(e) > 0$, (iii) $d < -\frac{b}{2a} < e$.

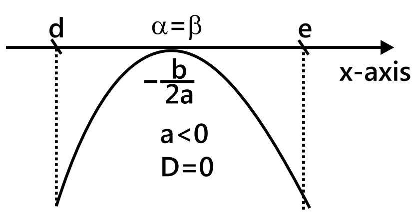


Figure 2.33

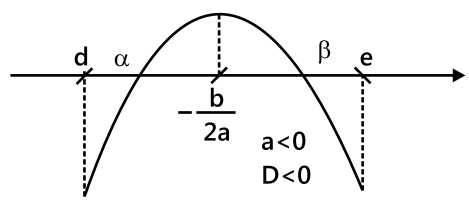


Figure 2.34

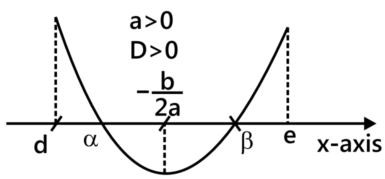


Figure 2.35

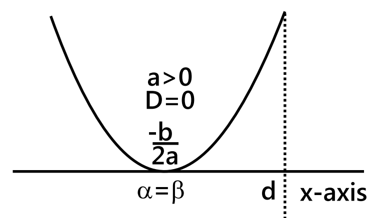


Figure 2.36

Type VI: One root is greater than e and the other root is less than 'd'.

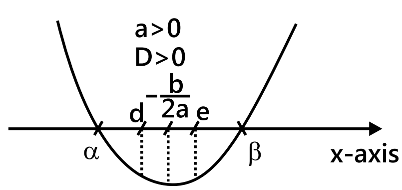


Figure 2.37

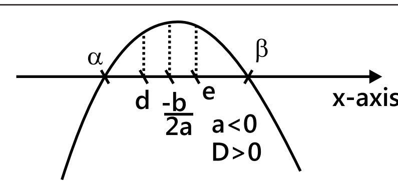


Figure 2.38

14. QUADRATIC EXPRESSION IN TWO VARIABLES

The general quadratic expression $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ can be factorized into two linear factors. The corresponding quadratic equation is in two variables

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

or $ax^2 + 2(hx + g)x + by^2 + 2fy + c = 0$ ...(i)

$$\therefore x = \frac{-2(hy+g) \pm \sqrt{4(hy+g)^2 - 4a(by^2+2fy+c)}}{2a} \Rightarrow x = \frac{-(hy+g) \pm \sqrt{h^2y^2+g^2+2ghy-2afy-ac-aby^2}}{a}$$

$$\Rightarrow ax+hy+g = \pm\sqrt{h^2y^2+g^2+2ghy-aby^2-2afy-ac} \quad .....(ii)$$

At this point, the expression (i) can be resolved into two linear factors if

$(h^2-ab)y^2+2(gh-af)y+g^2-ac$ is a perfect square and $h^2 - ab > 0$.

But $(h^2-ab)y^2+2(gh-af)y+g^2-ac$ will be a perfect square if D = 0

$\Rightarrow g^2h^2+a^2f^2-2afgh-h^2g^2+abg^2+ach^2-a^2bc=0$ and $h^2-ab>0$

$\Rightarrow abc+2fgh-af^2-bg^2-ch^2=0$ and $h^2-ab>0$

This is the required condition. The condition that this expression may be resolved into two linear rational factors is

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$\Rightarrow abc+2fgh-af^2-bg^2-ch^2=0$ and $h^2-ab>0$

This expression is called the discriminant of the above quadratic expression.

Illustration 48: If the equation $x^2 + 16y^2 - 3x + 2 = 0$ is satisfied by real values of x & y, then prove that

$x \in [1, 2]$, $y \in \left[\frac{-1}{8}, \frac{1}{8}\right]$ **(JEE MAIN)**

Sol: For real values of x and y, $D \geq 0$. Solve this by taking the x term and the y term constant one by one.

$x^2-3x+16y^2+2=0$; $D \geq 0$ as $x \in R$

$\Rightarrow 9-4(16y^2+2) \geq 0$; $\Rightarrow 9 - 64y^2 - 8 \geq 0$

$\therefore 64y^2-1 \leq 0$

$\Rightarrow (8y - 1)(8y + 1) \leq 0$ $\therefore y \in \left[\frac{-1}{8}, \frac{1}{8}\right]$

To find the range of x, in $16y^2+x^2-3x+2=0$ $D \geq 0$

Hence, $-64(x^2-3x+2) \geq 0$

Solving this, we get $x \in [1, 2]$

Illustration 49: Show that in the equation $x^2 - 3xy + 2y^2 - 2x - 3y - 35 - 0$, for every real value of x there is a real value of y. **(JEE MAIN)**

Sol: By using the formula $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ we will get $x = \frac{3y+2+\sqrt{y^2+24y+144}}{2}$.

Here, the quadratic equation in y is a perfect square.

$$x^2 - x(3y+2) + (2y^2 - 3y - 35) = 0$$

Now, x = 3y + 2 + 2$\sqrt{\text{quadratic in y}}$. As the quadratic equation in y is a perfect square$\left((y+12)^2\right)$.

$\therefore$ The relation between x & y is a linear equation which is a straight line.

$\therefore \forall x \in R$, y is a real value.

Illustration 50: If $(a_1x^2 + b_1x + c_1)y + (a_2x^2 + b_2x + c_2) = 0$ find the condition that x is a rational function of y **(JEE ADVANCED)**

Sol: For x is a rational function of y, its discriminant will be greater than or equal to zero, i.e. D ≥ 0.

$$x^2 - (a_1y + a_2) + x(b_1y + b_2) + (c_1y + c_2) = 0$$

$$x = \frac{-(b_1y+b_2) \pm \sqrt{(b_1y+b_2)^2 - 4(a_1y+a_2)(c_1y+c_2)}}{2(a_1y+a_2)}$$

For the above relation to exist $(b_1y+b_2)^2 - 4(a_1y+a_2)(c_1y+c_2) \geq 0$

$$\Rightarrow (b_1^2 - 4a_1c_1)y^2 + 2(b_1b_2 - 2a_1c_2 - 2a_2c_1)y + (b_2^2 - 4a_2c_2) \geq 0$$

$$\Rightarrow b_1^2 - 4a_1c_1 > 0 \text{ and } D \leq 0$$

Solving this will result in a relation for which x is a rational function of y.

15. NUMBER OF ROOTS OF A POLYNOMIAL EQUATION

(a) If f(x) is an increasing function in [a, b], then f(x) = 0 will have at most one root in [a, b].

(b) Let f(x) = 0 be a polynomial equation. a, b are two real numbers. Then f(x) =0 will have at least one real root or an odd number of real roots in (a, b) if f(a) and f(b) (a < b) are of opposite signs.

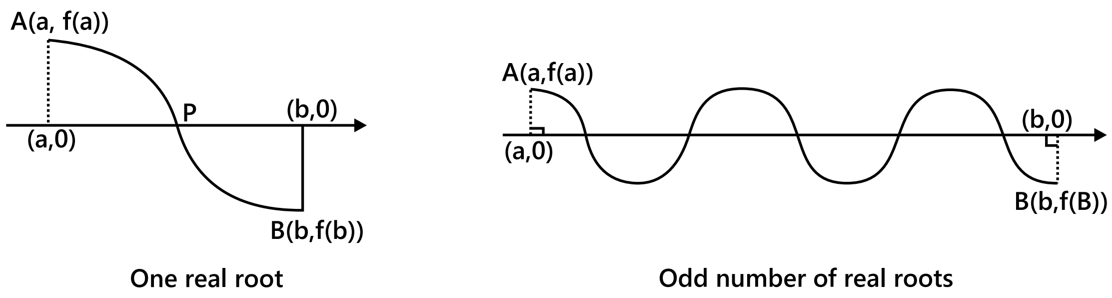


Figure 2.39

But if f(a) and f(b) are of the same sign, then either f(x) = 0 have one real root or an even number of real roots in (a, b)

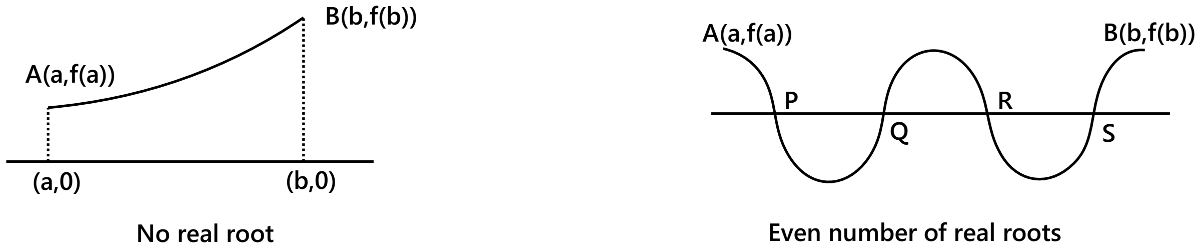


Figure 2.40

(c) If the equation f(x) = 0 has two real roots a and b, then f′(x) = 0 will have at least one real root lying between a and b (using Rolle's theorem).

MASTERJEE CONCEPTS

Descartes' rule of sign for the roots of a polynomial

Rule 1: The maximum number of positive real roots of a polynomial equation

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$ is the number of changes of the sign of coefficients from positive to negative and negative to positive. For instance, in the equation $x^3 + 3x^2 + 7x - 11 = 0$ the sign of the coefficients are +++- as there is just one change of sign, the number of positive roots of $x^3 + 3x^2 + 7x - 11 = 0$ is at most 1.

Rule 2 : The maximum number of negative roots of the polynomial equation f(x) = 0 is the number of changes from positive to negative and negative to positive in the sign of the coefficient of the equation f(–x) = 0.

Shivam Agarwal (JEE 2009 AIR 27)

PROBLEM-SOLVING TACTICS

Some hints for solving polynomial equations:

(a) To solve an equation of the form $(x-a)^4 + (x-b)^4 = A$; Put y = $x - \frac{a+b}{2}$

In general to solve an equation of the form $(x-a)^{2n} + (x-b)^{2n} = A$, where $n \in Z^+$, put y = $x - \frac{a+b}{2}$

(b) To solve an equation of the form, $a_0f(x)^{2n} + a_1(f(x))^n + a_2 = 0$ we put $(f(x))^n = y$ and solve $a_0y^2 + a_1y + a_2 = 0$ to obtain its roots y_1 and y_2.

Finally, to obtain the solution of (1) we solve, $(f(x))^n = y_1$ and $(f(x))^n = y_2$

(c) An equation of the form $(ax^2 + bx + c_1)(ax^2 + bx + c_2).....(ax^2 + bx + c_n) = A$. Where $c_1, c_2, ...c_n, A \in R$, can be solved by putting $ax^2 + bx = y$.

(d) An equation of the form (x – a)(x – b)(x – c)(x – d) = ⇒ Awhere ab = cd, can be reduced to a product of two quadratic polynomials by putting $y = x + \frac{ab}{2}$.

(e) An equation of the form (x – a) (x – b)(x – c)(x – d) = A where a < b < c < d, b – a = d – c can be solved by a change of variable $y = \frac{(x-a)+(x-b)+(x-c)+(x-d)}{4} = x - \frac{1}{4}(a+b+c+d)$

(f) A polynomial f(x, y) is said to be symmetric if f(x, y) = f(y, x) $\forall$ x, y. A symmetric polynomial can be represented as a function of x + y and xy.

Solving equations reducible to quadratic

(a) To solve an equation of the type $ax^4 + bx^2 + c = 0$, put $x^2 = y$.

(b) To solve an equation of the type $a(p(x))^2 + bp(x) + c = 0$ (p(x) is an expression of x), put p(x) = y.

(c) To solve an equation of the type $ap(x) + \frac{b}{p(x)} + c = 0$ where p(x) is an expression of x, put p(x)= y

This reduces the equation to $ay^2 + cy + b = 0$

(d) To solve an equation of the form $a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$, put $x + \frac{1}{x} = y$

and to solve $a\left(x^2 + \frac{1}{x^2}\right) + b\left(x - \frac{1}{x}\right) + c = 0$, put $x - \frac{1}{x} = y$

(e) To solve a reciprocal equation of the type $ax^4 + bx^3 + cx^2 + bx + a = 0,\ a \neq 0,$

we divide the equation by $\frac{d^2y}{dx^2}$ to obtain $a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$, and then put $x + \frac{1}{x} = y$

(f) To solve an equation of the type (x + a)(x + b)(x + c)(x + d) + k = 0 where a+b =c+d, put x^2 +(a+ b)x = y

(g) To solve an equation of the type $\sqrt{ax+b} = cx + d$ or $\sqrt{ax^2 + bx + c} = dx + e$, square both the sides.

(h) To solve an equation of the type $= \sqrt{ax+b} \pm \sqrt{cx+d} = e$, proceed as follows.

Step 1: Transfer one of the radical to the other side and square both the sides.

Step 2: Keep the expression with radical sign on one side and transfer the remaining expression on the other side

Step 3: Now solve as in 7 above.

FORMULAE SHEET

(a) **A quadratic equation is represented as :** $ax^2 + bx + c = 0,\ a \neq 0$

(b) **Roots of quadratic equation:** $x = \frac{-b \pm \sqrt{D}}{2a}$, where D(discriminant) = $b^2 - 4ac$

(c) **Nature of roots:** **(i)** D > 0 ⇒ roots are real and distinct (unequal)

(ii) D = 0 ⇒ roots are real and equal (coincident)

(iii) D < 0 ⇒ roots are imaginary and unequal

(d) The roots $(\alpha+i\beta)$, $(\alpha-i\beta)$ and $(\alpha+\sqrt{\beta}),(\alpha-\sqrt{\beta})$ are the conjugate pair of each other.

(e) **Sum and Product of roots :** If α and β are the roots of a quadratic equation, then

(i) $S=\alpha+\beta=\frac{-b}{a}=\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$ **(ii)** $P=\alpha\beta=\frac{c}{a}=\frac{\text{constant term}}{\text{Coefficient of } x^2}$

(f) **Equation in the form of roots:** $x^2-(\alpha+\beta)x+(\alpha.\beta)=0$

(g) In equation $ax^2+bx+c=0,\ a\neq 0$ If

(i) b = 0 $\Rightarrow$ roots are of equal magnitude but of opposite sign.

(ii) c = 0 $\Rightarrow$ one root is zero and other is –b/a

(iii) b = c = 0 $\Rightarrow$ both roots are zero.

(iv) a = c $\Rightarrow$ roots are reciprocal to each other.

(v) a > 0, c < 0 or a < 0, c > 0 $\Rightarrow$ roots are of opposite signs.

(vi) a > 0, b > 0, c > 0 or a < 0, b < 0, c < 0 $\Rightarrow$ both roots are –ve.

(vii) a > 0, b < 0, c > 0 or a < 0,b > 0, c < 0 $\Rightarrow$ both roots are +ve.

(h) The equations $a_1x^2+b_1x+c_1=0$ and $a_2x^2+b_2x+c_2=0$ have

(i) One common root if $\frac{b_1c_2-b_2c_1}{c_1a_2-c_2a_1}=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$

(ii) Both roots common if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

(i) In equation $ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a^2}\right]$

(i) If a>0, the equation has minimum value $=\frac{4ac-b^2}{4a}$ at $x=\frac{-b}{2a}$ and there is no maximum value.

(ii) If a < 0, the equation has maximum value $\frac{4ac-b^2}{4a}$ at $x=\frac{-b}{2a}$ and there is no minimum value.

(j) For cubic equation $ax^3+bx^2+cx+d=0$,

(i) $\alpha+\beta+\gamma=\frac{-b}{a}$

(ii) $\alpha\beta+\beta\gamma+\lambda\alpha=\frac{c}{a}$

(iii) $\alpha\beta\gamma=\frac{-d}{a}$ ... where α,β,γ are its roots.