8. SIMPLE HARMONIC MOTION AND ELASTICITY

SIMPLE HARMONIC MOTION

1. INTRODUCTION

There are so many examples of oscillatory or vibrational motion in our world. E.g. the vibrations of strings in a guitar or a sitar, the vibrations in the speakers of a music system, the to and fro motion of a pendulum, vibration in a suspension bridge as a vehicle passes on it, the oscillations in a tall building during an earthquake etc. Simple harmonic motion (SHM) is a type of oscillatory or vibrational motion. Every kind of oscillation or vibration of a particle or a system is not necessarily simple harmonic. The particle executing SHM like any other oscillatory motion has a variable acceleration, but this variation is different in different kinds of oscillations. The study of SHM is very useful and forms an important tool in understanding the characteristics of sound and light waves and alternating currents. Any oscillatory motion which is not simple harmonic can be expressed as a superposition of several simple harmonic motions of different frequencies.

2. PERIODIC AND OSCILLATORY MOTION

Periodic Motion: A motion which repeats itself after equal intervals of time is called periodic motion.

Oscillatory Motion: A body is said to possess oscillatory or vibratory motion if it moves back and forth repeatedly about a mean position. For an oscillatory motion, a restoring force is required.

Examples of Periodic and Oscillatory motion are revolution of earth around sun and motion of bob of a simple pendulum respectively.

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All Oscillatory motions are periodic but all Periodic motions need not be oscillatory.

A body experiencing force $F = -k(x-a)^n$ is in Oscillatory motion only if n is odd and its mean position is x=a. As, if n is odd only then we would have restoring force.

Vaibhav Krishan (JEE 2009, AIR 22)

2.1 Periodic Functions

A function is said to be periodic if it repeats itself after time period T i.e. the same function is obtained when the variable t is changed to t + T. Consider the following periodic functions:

$$f(t) = \sin\frac{2\pi}{T}t \quad \text{and} \quad g(t) = \cos\frac{2\pi}{T}t$$

Here T is the time period of the periodic motion. We shall see that if the variable t is changed to t + T, the same function results.

$$f(t+T) = \sin\left[\frac{2\pi}{T}(t+T)\right] = \sin\left[\frac{2\pi t}{T} + 2\pi\right] = \sin\left(\frac{2\pi t}{T}\right) \qquad \therefore\ f(t+T) = f(t)$$

Similarly, $g(t+T) = g(t)$

It can be easily verified that: $f(t+nT) = f(t)$ and $g(t+nT) = g(t)$

where n = 1,2,3,........

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These functions could be used to represent periodic motion i.e. Periodic functions represent periodic motion

T is the period of the above function.

To find periodicity of summation of two or more periodic functions the periodicity would be the L.C.M of the periodicities of the each function

Vaibhav Gupta (JEE 2009, AIR 54)

Illustration 1: Find the period of the function, $y = \sin\omega t + \sin 2\omega t + \sin 3\omega t$ **(JEE MAIN)**

Sol: The function with least angular frequency will have highest time period.

The given function can be written as, $y = y_1 + y_2 + y_3$

Here $y_1 = \sin\omega t$, $T_1 = \frac{2\pi}{\omega}$ $y_2 = \sin 2\omega t$, $T_2 = \frac{2\pi}{2\omega} = \frac{\pi}{\omega}$, and $y_3 = \sin 3\omega t$

$T_3 = \frac{2\pi}{3\omega}$ $\therefore T_1 = 2T_2$ and $T_1 = 3T_3$

So, the time period of the given function is T_1 or $\frac{2\pi}{\omega}$.

Because in time $T = \frac{2\pi}{\omega}$, first function completes one oscillation, the second function two oscillations and the third, three.

3. SIMPLE HARMONIC MOTION

Simple Harmonic Motion is a periodic motion in which a body moves to and fro about its mean position such that its restoring force or its acceleration is directly proportional to the displacement from its mean position and is directed towards its mean position. It can be expressed mathematically as, $F = m\frac{d^2x}{dt^2} = -kx.$ Where m is the mass on which a restoring force F acts to impart an acceleration $\frac{d^2x}{dt^2}$ along x-axis such that the restoring force F or acceleration is directly proportional to the displacement x along x-axis and k is a constant. The negative sign shows that the restoring force or acceleration is directed towards the mean position.

The differential equation of a simple harmonic motion is given by, $\frac{d^2x}{dt^2}+\left(\frac{K}{m}\right)x=0$ or $\frac{d^2x}{dt^2}+\omega^2x=0$

Where $\omega=\sqrt{\frac{K}{m}}=\sqrt{\frac{\text{acceleration}}{\text{displacement}}}$

The time period T, to complete one complete cycle by a body undergoing simple harmonic motion is given by

$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{\text{acceleration}}{\text{displacement}}}=2\pi\sqrt{\frac{m}{K}}$$

3.1 Types of SHM

Two Types of Simple Harmonic Motion

(a) Linear SHM (b) Angular SHM

Important among all oscillatory motion is the simple harmonic motion. A particle executing linear simple harmonic motion oscillates in straight line periodically in such a way that the acceleration is proportional to its displacement from a fixed point (called equilibrium), and is always directed towards that point.

If a body describes rotational motion in such a way that the direction of its angular velocity changes periodically and the torque acting on is always directed opposite to the angular displacement and magnitude of the torque is directly proportional to the angular displacement, then its motion is called angular SHM.

4. REPRESENTATION OF SIMPLE HARMONIC MOTION

If a point mass m is moving with uniform speed along a circular path of radius a, it's projection on the diameter of the circle along y-axis represents its simple harmonic motion (see Fig. 8.1). $y=a\sin\omega t$

Where ω is the uniform angular velocity of the body of mass m along a circular path of radius a than ωt is angle covered by the radius in time t from the initial position A at $t=0$ to the position B. As $\angle AOB = \angle OBC = \omega t$, the foot of perpendicular from B to the diameter YOY′ gives the projection at the point C such that $y=OC$ is the projection of this body on the diameter and represents the displacement of the body executing SHM along y-axis. If the body does not start its motion from the point A but at a point A′ so that $\angle AOA'$ is the phase angle ϕ,

then $y=a\sin(\omega t\pm\phi)$ ... (i)

Where ϕ is the phase angle which may be positive or negative. The phase angle represents the fraction of the angle by which the motion of the body is out of step between the initial position of the body and the mean position of simple harmonic motion. The phase difference is the fraction of angle 2π or time period T of SHM by which the body is out of step initially from the mean position of the body.

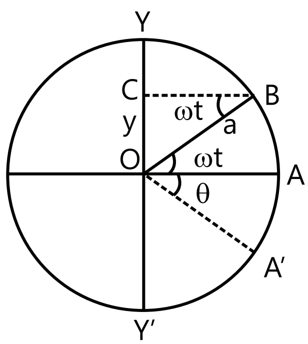

Figure 8.1: Particle moving in a circle with angular speed ω in X-Y plane

Differentiating equation (i), $\frac{dy}{dt}=v=a\omega\cos(\omega t\pm\phi)$

As $\sin(\omega t\pm\phi)=\frac{y}{a},\cos(\omega t\pm\phi)=\sqrt{1-\sin^2(\omega t\pm\phi)}=\sqrt{1-\frac{y^2}{a^2}}=\frac{\sqrt{a^2-y^2}}{a}$

$\therefore v=\omega\sqrt{a^2-y^2}$ ... (ii)

Differentiating (ii), the acceleration $=\frac{d^2y}{dt^2}=-a\omega^2\sin(\omega t\pm\phi)$

$$\therefore \frac{d^2y}{dt^2}=-\omega^2y$$

It represents the equation of simple harmonic motion where $\omega = \sqrt{\left(\frac{d^2y}{dt^2}\Big/ y\right)} = \sqrt{\frac{\text{acceleration}}{\text{displacement}}}$

Time period, $T = 2\pi/\omega = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$

4.1 Alternative Method for Finding Velocity and Acceleration in SHM

Let v be the velocity of the reference particle at P. Resolve velocity V into two rectangular components $V\cos\theta$ parallel to YOY′ and $V\sin\theta$ perpendicular to YOY′ (see Fig. 8.2). The velocity v of the projection N is clearly $V\cos\theta$.

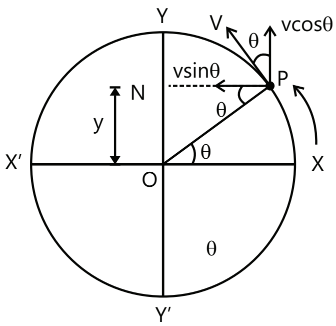

Figure 8.2: Relation between v and ω.

$\therefore\ v = V\cos\theta = A\omega\cos\omega t$ or $v = A\omega\sqrt{1-\sin^2\omega t}$

or $v = A\omega\sqrt{1-\frac{y^2}{A^2}}$ or $v = \cancel{A}\omega\sqrt{\frac{A^2-y^2}{\cancel{A^2}}}$ or $v = \omega\sqrt{A^2-y^2}$

The centripetal acceleration $\frac{V^2}{A}$ of the particle at P can be resolved into two rectangular components $-\frac{V^2}{A}\cos\theta$ Perpendicular to YOY′ and $\frac{V^2}{A}\sin\theta$ anti-parallel to YOY′ Acceleration of $N = -\frac{V^2}{A}\sin\theta$

or $\text{Acceleration} = -\frac{V^2}{A^2}(A\sin\theta) = -\omega^2(A\sin\omega t)$

or $\text{Acceleration} = -\omega^2 y$

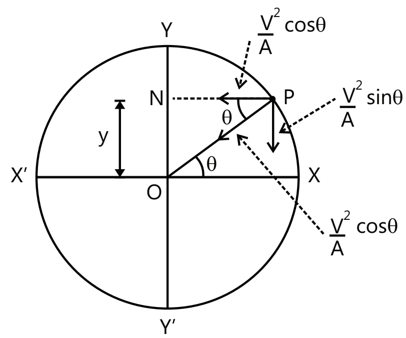

Figure 8.3: Direction of centripetal acceleration of particle

4.2 Time Period or Periodic Time of SHM

It is the smallest interval of time at which the details of motion repeat. It is generally represented by T.

$$'x'\text{ at }(t+T) = A\cos\left(2\pi\frac{t+T}{T}+\phi_0\right) = A\cos\left(\frac{2\pi t}{T}+2\pi+\phi_0\right) \quad \text{... (i)}$$

It is clear from here that the details of motion repeat after time T. Time period may also be defined as the time taken by the oscillating particle to complete one oscillation. It is equal to the time taken by the reference particle to complete one revolution. In one revolution, the angle traversed by the reference particle is 2π radian and T is the time taken. If ω be the uniform angular velocity of the reference particle, then $\omega = \frac{2\pi}{T}$ or $T = \frac{2\pi}{\omega}$

4.3 Frequency

It is the number of oscillations (or vibrations) completed per unit time. It is denoted by f. In time T second, one vibration is completed.

In 1 second, $\frac{1}{T}$ vibrations are completed   or $f = \frac{1}{T}$ or $fT = 1$

Also, $\omega = \frac{2\pi}{T} = 2\pi\times\frac{1}{T} = 2\pi f$ So, equation (i) may also be written as under

$$x = A\cos(2\pi ft + \phi_0) \quad \text{... (ii)}$$

The unit of f is s^{-1} or hertz or 'cycles per second' (cps). $\therefore \; \phi s^{-1} = \phi Hz = \phi cps$.

4.4 Angular Frequency

It is frequency f multiplied by a numerical quantity 2π. It is denoted by ω so that $\omega = 2\pi f = \frac{2\pi}{T}$. Equation (vi) may be written as $x = A\cos(\omega t + \phi)$

4.5 Phase

Phase of a vibrating particle at any instant is the state of the vibrating particle regarding its displacement and direction of vibration at that particular instant.

The argument of the cosine in equation $x = A\cos(\omega t + \phi_0)$ gives the phase of oscillation at time t.

It is denoted by ϕ. $\therefore \; \phi = 2\pi\frac{t}{T} + \phi_0$ or $\phi = \omega t + \phi_0$

It is clear that phase ϕ is a function of time t. The phase of a vibrating particle can be expressed in terms of fraction of the time period that has elapsed since the vibrating particle left its initial position in the positive direction. Again, $\phi - \phi_0 = \omega t = \frac{2\pi t}{T}$. So, the phase change in time t is $\frac{2\pi t}{T}$. The phase change in T second will be 2π which actually means a 'no change in phase'. Thus, time period may also be defined as the time interval during which the phase of the vibrating particle changes by 2π.

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The phase difference between acceleration and displacement is 180°. In SHM phase difference between velocity and acceleration is $\pi/2$ and velocity and displacement is $\pi/2$.

Fig. 8.4 (a) displacement, (b) velocity and (c) acceleration vs. time in SHM.

$v = \pm\omega\sqrt{A^2 - y^2}$. Graphical variation of v with y is an ellipse.

Max velocity at $y = 0$ i.e. at mean position and

$V_{max} = A\omega$; $a = -\omega^2 y$

Graph between acceleration and displacement of a particle executing SHM is straight line.

Max acceleration at $y = A$ i.e. at extreme position and $a_{max} = A\omega^2$

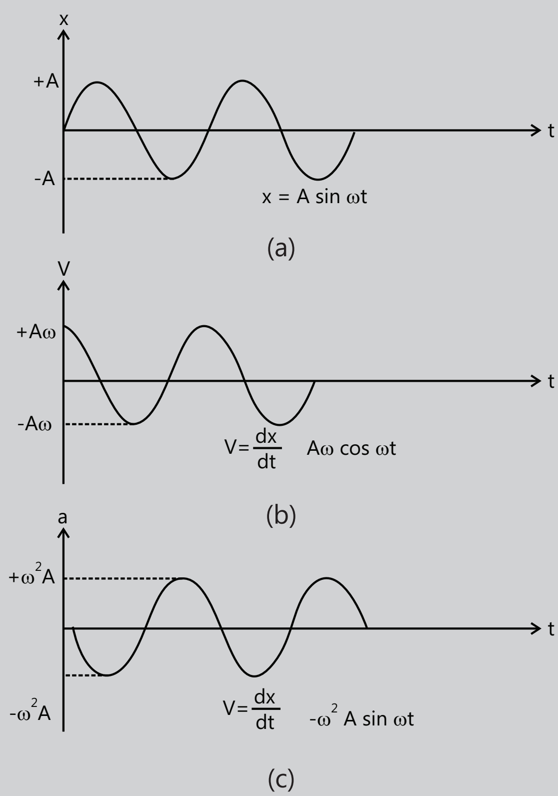

Figure 8.4

Nivvedan (JEE 2009, AIR 113)

Illustration 2: A particle executes simple harmonic motion about the point $x = 0$. At time $t = 0$ it has displacement $x = 2$ cm and zero velocity. If the frequency of motion is $0.25\,s^{-1}$, find (a) the period, (b) angular frequency, (c) the amplitude, (d) maximum speed, (e) the displacement at $t = 3s$ and (f) the velocity at $t = 3s$. **(JEE MAIN)**

Sol: The standard equation for displacement in SHM is $x = A\sin(\omega t + \phi)$. When velocity is zero, the particle is at maximum displacement.

(a) Period $T = \frac{1}{f} = \frac{1}{0.25 s^{-1}} = 4s$

(b) Angular frequency $\omega = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}rad/s = 1.57 rad/s$

(c) Amplitude is the maximum displacement from mean position. Hence, $A = 2 - 0 = 2cm$

(d) Maximum speed $v_{max} = A\omega = 2.\frac{\pi}{2} = \pi cm/s = 3.14\ cm/s$

(e) The displacement is given by $x = A\sin(\omega t + \phi)$

Initially at t=0; $x = 2cm$, then $2 = 2\sin\phi$ or $\sin\phi = 1 = \sin 90^\circ$ or $\phi = 90^\circ$

Now, at $t = 3s$ $x = 2\sin\left(\frac{\pi}{2}\times 3 + \frac{\pi}{2}\right) = 0$

(f) Velocity at $x = 0$ is v_{max} i.e.,3.14 cm/s.

Illustration 3: Two particles move parallel to x-axis about the origin with the same amplitude and frequency. At a certain instant, they are found at distance $\frac{A}{3}$ from the origin on opposite sides but their velocities are found to be in the same direction. What is the phase difference between the two? **(JEE ADVANCED)**

Sol: The standard equation for displacement in SHM is $x = A\sin(\omega t + \phi)$. Displacement on opposite sides of the mean position has opposite signs. Equation for velocity is $v = A\omega\cos(\omega t + \phi)$. Velocities in same direction have same sign.

Let equations of two SHM be $x_1 = A\sin\omega t$ ... (i)

$x_2 = A\sin(\omega t + \phi)$ ... (ii)

Give that $\frac{A}{3} = A\sin\omega t$ and $-\frac{A}{3} = A\sin(\omega t + \phi)$ Which gives $\sin\omega t = \frac{1}{3}$ ... (iii)

$\sin(\omega t + \phi) = -\frac{1}{3}$ ... (iv)

From Eq.(iv), $\sin\omega t\cos\phi + \cos\omega t\sin\phi = -\frac{1}{3}$; $\frac{1}{3}\cos\phi + \sqrt{1 - \frac{1}{9}}\sin\phi = -\frac{1}{3}$

Solving this equation, we get or $\cos\phi = -1, \frac{7}{9}$; $\phi = \pi$ or $\cos^{-1}\left(\frac{7}{9}\right)$

Differentiating Eqs. (i) and (ii), we obtain; $v_1 = A\omega\cos\omega t$ and $v_2 = A\omega\cos(\omega t + \phi)$

If we put $\phi = \pi$, we find v_1 and v_2 are of opposite signs. Hence, $\phi = \pi$ is not acceptable.

$\phi = \cos^{-1}\left(\frac{7}{9}\right)$

Illustration 4: With the assumption of no slipping, determine the mass m of the block which must be placed on the top of a 6 kg cart in order that the system period is 0.75s. What is the minimum coefficient of static fraction μ_s for which the block will not slip relative to the cart if the cart is displaced 50mm from the equilibrium position and released? Take $\left(g = 9.8 m/s^2\right)$. **(JEE ADVANCED)**

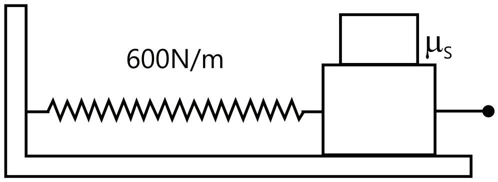

Figure 8.5

Sol: $\omega = \sqrt{\frac{k}{M}}$ where M is the total mass attached to the spring. The maximum restoring force on the blocks will be at the extreme position. The limiting friction on mass m should be greater than or equal to the maximum restoring force required for mass m.

(a) $T = 2\pi\sqrt{\frac{m+6}{600}}$ $\left(\because T = 2\pi\sqrt{\frac{m}{k}}\right)$ $\therefore$ $0.75 = 2\pi\sqrt{\frac{m+6}{600}}$; $m = \frac{(0.75)^2 \times 600}{(2\pi)^2} - 6 = 2.55\text{kg}$

(b) Maximum acceleration of SHM is $a_{max} = \omega^2 A$ $(A = \text{amplitude})$

i.e., maximum force on mass 'm' is $m\omega^2 A$ which is being provided by the force of friction between the mass and the cart. Therefore, $\mu_s mg \ge m\omega^2 A$ or $\mu_s \ge \frac{\omega^2 A}{g}$ or $\mu_s \ge \left(\frac{2\pi}{T}\right)^2 \frac{A}{g}$ or $\mu_s \ge \left(\frac{2\pi}{0.75}\right)^2 \left(\frac{0.05}{9.8}\right)$ $(A = 50\text{mm})$

or $\mu_s \ge 0.358$. Thus, the minimum value of μ_s should be 0.358.

5. ENERGY IN SHM

The displacement and the velocity of a particle executing a simple harmonic motion are given by

$x = A\sin(\omega t + \delta)$ and $v = A\omega\cos(\omega t + \delta)$. The potential energy at time t is, therefore,

$U = \frac{1}{2}kx^2$ and $k = m\omega^2$ Therefore $U = \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \delta)$, and the kinetic energy at time t is $K = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\cos^2(\omega t + \delta)$

The total mechanical energy time t is $E = U + K$

$= \frac{1}{2}m\omega^2 A^2\left[\sin^2(\omega t + \delta) + \left(\cos^2(\omega t + \delta)\right)\right] = \frac{1}{2}m\omega^2 A^2$ Average value of P.E. and K.E

By equation (i) P.E. at distance x is given by

$U = \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi)$ $\{\text{since at time t}, x = A\sin(\omega t + \phi)\}$

The average value of P.E. of complete vibration is given by

$$U_{average} = \frac{1}{T}\int_0^T U dt = \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) = \frac{m\omega^2 A^2}{4T}\int_0^T 2\sin^2(\omega t + \phi)dt = \frac{1}{4}m\omega^2 A^2$$

Because the average value of sine square or cosine square function for the complete cycle is 0.

Now, KE at x is given by $K.E. = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}m\left[\frac{d}{dt}\{A\sin(\omega t + \phi)\}\right]^2 = \frac{1}{2}m\omega^2 A^2\cos^2(\omega t + \phi)$

The average value of K.E. for complete cycle $K.E._{average} = \frac{1}{T}\int_0^T \frac{1}{2} m\omega^2 A^2 \cos^2(\omega t + \phi)\,dt$

$$= \frac{m\omega^2 A^2}{4T}\int_0^T \{1 + \cos 2(\omega t + \phi)\}\,dt = \frac{m\omega^2 A^2}{4T}.\ T = \frac{1}{4} m\omega^2 A^2$$

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Thus average values of K.E. and P.E. of harmonic oscillator are equal to half of the total energy.

The total mechanical energy is constant but the kinetic energy and potential energy of the particle are oscillating

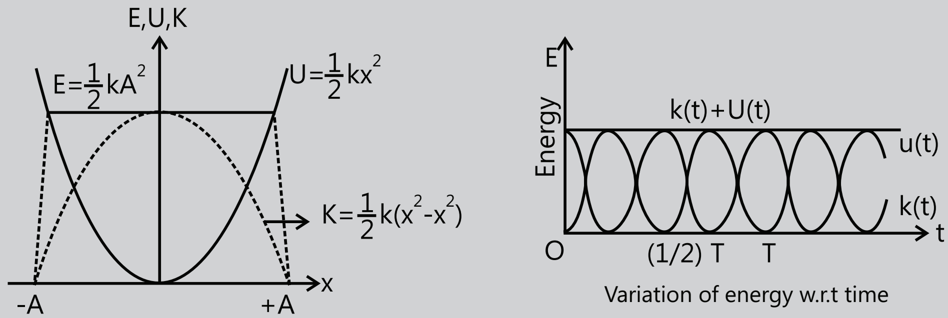

Figure 8.6

Graph for Energy of SHM: Figure 8.6 shows the variation of total energy (E), Potential energy (U) and kinetic energy (K) with Displacement (x).

Chinmay S Purandare (JEE 2012, AIR 698)

At a glance

S.No.	Name of the equation	Expression of the Equation	Remarks
1.	Displacement-time	$x = A\cos(\omega t + \phi)$	X varies between +A and –A
2.	Velocity-time $\left(V = \frac{dx}{dt}\right)$	$v = -A\omega\sin(\omega t + \phi)$	v varies between $+A\omega$ and $-A\omega$
3.	Acceleration-time $\left(a = \frac{dv}{dt}\right)$	$a = -A\omega^2\cos(\omega t + \phi)$	a varies between $+A\omega^2$ and $-A\omega^2$
4.	Kinetic energy-time $\left(K = \frac{1}{2}mv^2\right)$	$K = \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \phi)$	K varies between 0 and $\frac{1}{2}mA^2\omega^2$
5.	Potential energy-time $\left(U = \frac{1}{2}m\omega^2 x^2\right)$	$K = \frac{1}{2}m\omega^2 A^2\cos^2(\omega t + \phi)$	U varies between $\frac{1}{2}mA^2\omega^2$ and 0
6.	Total energy-time $(E = K + U)$	$E = \frac{1}{2}m\omega^2 A^2$	E is constant

S.No.	Name of the equation	Expression of the Equation	Remarks
7.	Velocity-displacement	$v = \omega\sqrt{A^2 - X^2}$	$v = 0$ at $x = \pm A$ and at $x = 0$ $v = \pm A\omega$
8.	Acceleration-displacement	$a = -\omega^2 x$	$a = 0$ at $x = 0$ $a = \pm\omega^2 A$ at $x = \mp A$
9.	Kinetic energy-displacement	$K = \frac{1}{2}m\omega^2\left(A^2 - X^2\right)$	$K = 0$ at $x = \mp A$ $K = \frac{1}{2}m\omega^2 A^2$ at $x = 0$
10.	Potential energy-displacement	$U = \frac{1}{2}m\omega^2 x^2$	$U = 0$ at $x = 0$ $U = \frac{1}{2}m\omega^2 A^2$ at $x = \pm A$
11.	Total energy-displacement	$E = \frac{1}{2}m\omega^2 A^2$	E is constant

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At mean position $\rightarrow$ K is the maximum and U is the minimum (it may be zero also, but it is not necessarily zero). At extreme positions $\rightarrow$ K is zero and U is the maximum.

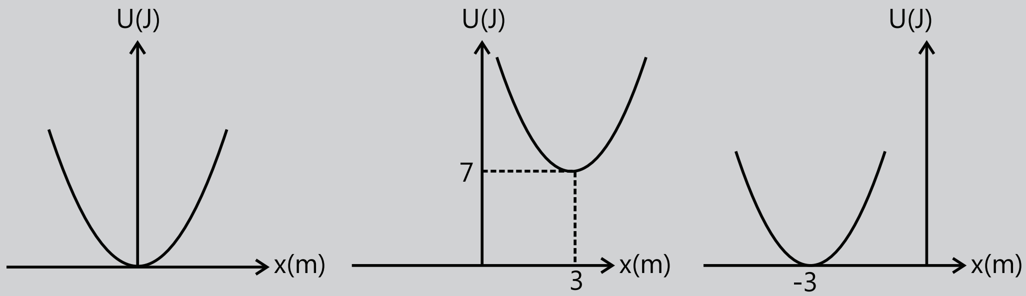


Figure 8.7

B Rajiv Reddy (JEE 2012, AIR 11)

Illustration 5: The potential energy of a particle oscillating on x-axis is given as $U = 20 + \left(x - 2\right)^2$

Here, U is in joules and x in meters. Total mechanical energy of the particle is 36J. **(JEE MAIN)**

(a) State whether the motion of the particle is simple harmonic or not.

(b) Find the mean position.

(c) Find the maximum kinetic energy of the particle.

Sol: At the mean position the kinetic energy is the maximum and potential energy is the minimum. The sum of kinetic energy and potential energy is constant throughout the SHM, equal to the total mechanical energy.

(a) $F = -\frac{dU}{dx} = -2\left(x - 2\right)$

By assuming $x-2=X$, we have $F=-2x$, Since, $F \propto -X$

The motion of the particle is simple harmonic

(b) The mean position of the particle is $X=0$ or $x-2=0$, which gives $x=2\,m$

(c) Maximum kinetic energy of the particle is, $K_{max}=E-U_{min}=36-20=16J$

Note U_{min} is 20J at mean position or at $x=2m$.

Illustration 6: A block with mass M attached to a horizontal spring with force constant k is moving with simple harmonic motion having amplitude A_1. At the instant when the block passes through its equilibrium position a lump of putty with mass m is dropped vertically on the block from a very small height and sticks to it.

(JEE ADVANCED)

(a) Find the new amplitude and period.

(b) Repeat part (a) for the case in which the putty is dropped on the block when it is at one end of its path.

Sol: Sticking of putty constitutes an inelastic collision. Kinetic energy at equilibrium position converts into potential energy at extreme position, $\frac{1}{2}mv^2=\frac{1}{2}kA^2$.

(a) Before the lump of putty is dropped the total mechanical energy of the block and spring is $E_1=\frac{1}{2}kA_1^2$. Since, the block is at the equilibrium position, $U=0$, and the energy is purely kinetic. Let v_1 be the speed of the block at the equilibrium position, we have $v_1=\sqrt{\frac{k}{M}}A_1$

During the process momentum of the system in horizontal direction is conserved. Let v_2 be the speed of the combined mass, then $(M+m)v_2=Mv_1$; $v_2=\frac{M}{M+m}v_1$

Now, let A_2 be the amplitude afterwards. Then, $E_2=\frac{1}{2}kA_2^2=\frac{1}{2}(M+m)v_2^2$

Substituting the proper values, we have $A_2=A_1\sqrt{\frac{M}{M+m}}$

Note: $E_2<E_1$, as some energy is lost into heating up the block and putty. Further, $T_2=2\pi\sqrt{\frac{M+m}{k}}$

(b) When the putty drops on the block, the block is instantaneously at rest. All the mechanical energy is stored in the spring as potential energy. Again the momentum in horizontal direction is conserved during the process, but now it is zero just before and after putty is dropped. So, in this case, adding the extra mass of the putty has no effect on the mechanical energy, i.e.,

$E_2=E_1=\frac{1}{2}kA_1^2$ and the amplitude is still A_1. Thus, $A_2=A_1$ and $T_2=2\pi\sqrt{\frac{M+m}{k}}$

6. ANGULAR SIMPLE HARMONIC MOTION

When a particle executes SHM on a curve path, then it is said to be angular SHM. E.g. - Simple pendulum. In this case, to find out the time period, we find out restoring torque and hence angular acceleration.

i.e. $\tau=-k\theta$ Where k is a constant $\Rightarrow$ $I\alpha=-k\theta$ where I is moment of inertia ... (i)

$$\Rightarrow \alpha=\frac{-k}{I}\theta \quad \text{... (ii)}$$

Also, the equation of SHM for angular SHM, is $\alpha=-\omega^2\theta$. Comparing (i) and (ii), we get ω, hence the time period.

Problem solving strategy

Step 1: Find the stable equilibrium position which usually is also known as the mean position. Net force or torque on the particle in this position is zero. Potential energy is the minimum.

Step 2: Displace the particle from its mean position by a small displacement x (in case of a linear SHM) or θ (in case of an angular SHM).

Step 3: Find net force or torque in this displaced position.

Step 4: Show that this force or torque has a tendency to bring the particle back to its mean position and magnitude of force or torque is proportional to displacement, i.e.,

$F \propto -x$ or $F = -kx$ ...(i); $\tau \propto -\theta$ or $\tau = -k\theta$ ...(ii)

This force or torque is also known as restoring force or restoring torque.

Step 5: Find linear acceleration by dividing Eq.(i) by mass m or angular acceleration by dividing Eq.(ii) by moment of inertia I.

Hence, $a = -\frac{k}{m}.x = -\omega^2 x$ or $\alpha = -\frac{k}{I}\theta = -\omega^2\theta$

Step 6: Finally, $\omega = \sqrt{\left|\frac{a}{x}\right|}$ or $\sqrt{\left|\frac{\alpha}{\theta}\right|}$ or $\frac{2\pi}{T} = \sqrt{\left|\frac{a}{x}\right|}$ or $\sqrt{\left|\frac{\alpha}{\theta}\right|}$

$\therefore\ T = 2\pi\sqrt{\left|\frac{x}{a}\right|}$ or $2\pi\sqrt{\left|\frac{\theta}{\alpha}\right|}$

Energy Method: Repeat step 1 and step 2 as in method 1. Find the total mechanical energy (E) in the displaced position. Since, mechanical energy in SHM remains constant. $\frac{dE}{dt} = 0$ By differentiating the energy equation with respect to time and substituting $\frac{dx}{dt} = v$, $\frac{d\theta}{dt} = \omega$, $\frac{dv}{dt} = a$, and $\frac{d\omega}{dt} = \alpha$ we come to step 5. The remaining procedure is same.

Note: (i) E usually consists of following terms:

(a) Gravitational PE (b) Elastic PE (c) Electrostatic PE (d) Rotational KE and (e) Translational KE

(ii) For gravitational PE, choose the reference point (h=0) at mean position.

Illustration 7: Calculate the angular frequency of the system shown in Fig 8.8. Friction is absent everywhere and the threads, spring and pulleys are massless. Given, that $m_A = m_B = m$. **(JEE ADVANCED)**

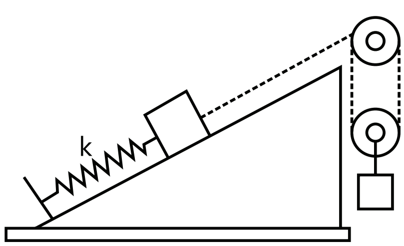


Figure 8. 8

Sol: This problem can be solved either by restoring force method or by the energy method. The gain in kinetic energy is at the cost of decrease in gravitational and/or elastic potential energy.

Let x_0 be the extension in the spring in equilibrium. Then equilibrium of A and B give $T = kx_0 + mg\sin\theta$ ... (i)

and $2T = mg$ ... (ii)

Here, T is the tension in the string. Now, suppose A is further displaced by a distance x from its mean position and v be its speed at this moment. Then B lowers by $\frac{x}{2}$ and speed of B at this instant will be $\frac{v}{2}$. Total energy of the system in this position will be,

$$E=\frac{1}{2}k\left(x+x_0\right)^2+\frac{1}{2}m_A v^2+\frac{1}{2}m_B\left(\frac{V}{2}\right)^2+m_A gh_A-m_B gh_B$$

or $E=\frac{1}{2}k\left(x+x_0\right)^2+\frac{1}{2}mv^2+\frac{1}{8}mv^2+mgx\sin\theta-mg\frac{x}{2}$

or $E=\frac{1}{2}k\left(x+x_0\right)^2+\frac{5}{8}mv^2+mgx\sin\theta-mg\frac{x}{2}$

Since, E is constant, $\frac{dE}{dt}=0$ or $0=k\left(x+x_0\right)\frac{dx}{dt}+\frac{5}{4}mv\left(\frac{dv}{dt}\right)+mg(\sin\theta)\left(\frac{dx}{dt}\right)-\frac{mg}{2}\left(\frac{dx}{dt}\right)$

Substituting, $\frac{dx}{dt}=v$; $\frac{dv}{dt}=a$ and $kx_0+mg\sin\theta=\frac{mg}{2}$ [From Eqs. (i) and (ii)]

We get, $\frac{5}{4}ma=-kx$ Since, $a\propto -x$

Motion is simple harmonic, time period of which is, $T=2\pi\sqrt{\left|\frac{x}{a}\right|}=2\pi\sqrt{\frac{5m}{4k}}$ $\therefore$ $\omega=\frac{2\pi}{T}=\sqrt{\frac{4k}{5m}}$

7. SIMPLE PENDULUM

It is an example of angular simple harmonic motion. Let's calculate its time period. Let us suppose that a bob of mass m is executing SHM (see Fig. 8.9). The length of the pendulum is ℓ, which is the distant between the point of oscillation and the center of mass of the bob. Torque acting on the bob about the point O.

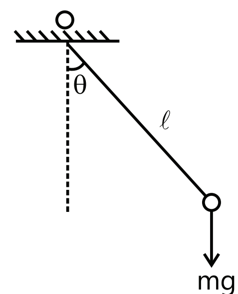


Figure 8.9: Oscillations of simple Pendulum

$\Gamma=mg\,\ell\sin\theta$ (And for small θ, $\sin\theta\simeq\theta$)

$\Rightarrow\Gamma=mg\ell\theta\ \Rightarrow I\alpha=-mg\ell\theta\ \Rightarrow\ \alpha=-\frac{mg\ell}{I}\theta$

where α is angular acceleration $=-\frac{mg\ell}{m\ell^2}\theta$; $\alpha=-\frac{g}{\ell}\theta$ ... (i)

The equation of SHM is $\alpha=-\omega^2\theta$ ... (ii)

Comparing (i) and (ii), we get $\omega^2=\frac{g}{\ell}$; $\omega=\sqrt{\frac{g}{\ell}}$ $\Rightarrow$ $\frac{2\pi}{T}=\sqrt{\frac{g}{\ell}}$; $T=2\pi\sqrt{\frac{\ell}{g}}$

MASTERJEE CONCEPTS

The period is independent of the mass of the suspended particle.

Nitin Chandrol (JEE 2012, AIR 134)

Various scenarios: Time period

Pendulum in a lift descending with acceleration "a", $T=2\pi\sqrt{\frac{\ell}{(g-a)}}$

Pendulum in a lift ascending with acceleration "a", $T=2\pi\sqrt{\frac{\ell}{(g+a)}}$

Pendulum suspended in a train accelerated with ''a" uniformly in horizontal direction $T=2\pi\sqrt{\frac{\ell}{\left(a^2+g^2\right)^{\frac{1}{2}}}}$

Pendulum suspended in car taking turn with velocity v in a circular path of radius r, $T = 2\pi\sqrt{\dfrac{\ell}{\left(\left(\dfrac{v^2}{r}\right)^2 + g^2\right)^{\frac{1}{2}}}}$

Note: If the pendulum is suspended in vacuum, then the time period of the pendulum decreases.

Illustration 8: A simple pendulum consists of a small sphere of mass m suspended by a thread of length ℓ. The sphere carries a positive charge q. The pendulum is placed in a uniform electric field of strength E directed vertically upwards. With what period will pendulum oscillate if the electrostatic force acting on the sphere is less than the gravitational force? **(JEE MAIN)**

Sol: The electrostatic force is acting opposite to the weight of the block. So the effective value of acceleration due to gravity will be less than the actual value of g.

The two forces acting on the bob are shown in Fig 8.10.

g_{eff} in this case will be $\dfrac{w - F_e}{m}$ or $g_{eff} = \dfrac{mg - qE}{m} = g - \dfrac{qE}{m}$

$\therefore\ T = 2\pi\sqrt{\dfrac{\ell}{g_{eff}}} = 2\pi\sqrt{\dfrac{\ell}{g - \dfrac{qE}{m}}}$

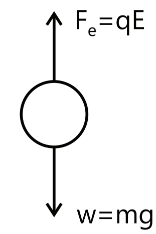


Figure 8.10

MASTERJEE CONCEPTS

In case of a pendulum clock, time is lost if T increase and gained if T decreases. Time lost or gained in time t is given by.

$\Delta t = \dfrac{\Delta T}{T'}t$ e.g., if $T = 2s, T' = 3s$, then $\Delta T = 1s$

$\therefore$ Time lost by the clock in 1 hr. $\Delta t = \dfrac{1}{3} \times 3600 = 1200s$

Second pendulum is a with its time period precisely 2 seconds

Vaibhav Gupta (JEE 2009, AIR 54)

Illustration 9: A simple pendulum of length l is suspended from the ceiling of a cart which is sliding without friction on an inclined plane of inclination θ. What will be the time period of the pendulum? **(JEE MAIN)**

Sol: The cart accelerates down the plane with acceleration a = g sinθ.

$g_{eff} = |\vec{g} - \vec{a}| = \sqrt{g^2 + 2g^2\sin\theta\cos(90^\circ + \theta) + g^2\sin^2\theta} = g\cos\theta$

Here, point of suspension has acceleration. $\vec{a} = g\sin\theta$ (down the Plane). Further, $\vec{g}$ can be resolved into two components g sin θ (along the plane) and g cos θ (perpendicular to plane)

$\therefore \vec{g}_{eff} = \vec{g} - \vec{a} = g\cos\theta$

(perpendicular to plane)

$\therefore T = 2\pi\sqrt{\dfrac{\ell}{|\vec{g}_{eff}|}} = 2\pi\sqrt{\dfrac{\ell}{g\cos\theta}}$

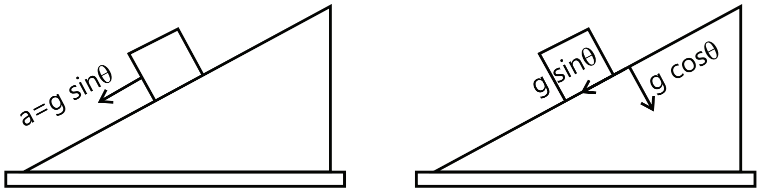


Figure 8.11

Note: If $\theta = 0^{\circ}, T = 2\pi\sqrt{\frac{\ell}{g}}$ which is quiet obvious.

8. PHYSICAL PENDULUM

Any rigid body mounted so that it can swing in a vertical plane about some axis passing through it is called a physical pendulum (see Fig. 8.12).

A body of irregular shape is pivoted about a horizontal frictionless axis through P and displaced from the equilibrium position by an angle θ. (The equilibrium position is that in which the center of mass C of the body lies vertically below P).

The distance from the pivot to the center of mass is d. The moment of inertia of the body about an axis through the pivot is I and the mass of the body is M. The restoring torque about the point P,

$\tau = Mgd\theta$ (if θ be very small, $\sin\theta = \theta$)

$\tau = Mgd\theta$; $I\alpha = -Mgd\theta$; $\alpha = -\frac{Mgd}{I}\theta$ ......(i)

Comparing with the equation of SHM

$\omega^2 = \frac{Mgd}{I}$; $\omega = \sqrt{\frac{Mgd}{I}}$; $2\pi / T = \sqrt{\frac{Mgd}{I}}$; $T = 2\pi\sqrt{\frac{I}{Mgd}}$

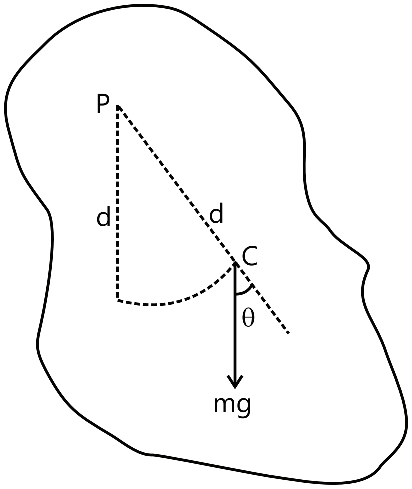


Figure 8.12: Rotation of Physical Pendulum

MASTERJEE CONCEPTS

It may be necessary to use parallel axis theorem to find Moment of Inertia about the pivoted axis $I = I_G + ml^2$

Yashwanth Sandupatla (JEE 2012, AIR 821)

9. TORSIONAL PENDULUM

In torsional pendulum, an extended body is suspended by a light thread or a wire (see Fig. 8.13). The body is rotated through an angle about the wire as the axis of rotation. The wire remains vertical during this motion but a twist is produced in the wire. The lower end of the wire is rotated through an angle with the body but the upper end remains fixed with the support. Thus, a twist θ is produced. The twisted wire exerts a restoring torque on the body to bring it back to its original position in which the twist θ in the wire is zero. This torque has a magnitude proportional to the angle of twist which is equal to the angle rotated by the body. The proportionality constant is called the torsional constant of the wire. Thus, if the torsional constant of the wire is κ and the body is rotated through an angle θ, the torque produced is $\Gamma = -\kappa\theta$. If I be the moment of inertia of the body about the vertical axis, the angular acceleration is $\alpha = \frac{\Gamma}{I} = -\frac{\kappa}{I}\theta = -\omega^2\theta$ where $\omega = \sqrt{\frac{\kappa}{I}}$

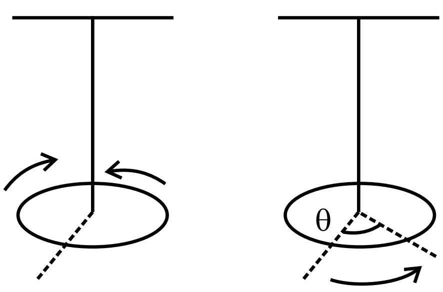


Figure 8.13: Torsional pendulum

Thus, the motion of the body is simple harmonic and the time period is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{\kappa}}$

Illustration 10: A ring of radius r is suspended from a point on its circumference. Determine its angular frequency of small oscillations. **(JEE ADVANCED)**

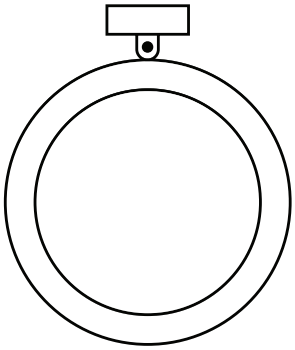


Figure 8.14

Sol: This is an example of a physical pendulum. Find moment of inertia about point of suspension and the distance of the point of suspension from the center of mass.

It is physical pendulum, the time period of which is, $T = 2\pi\sqrt{\frac{I}{mgl}}$

Here, I = moments of inertia of the ring about point of suspension

$= mr^2 + mr^2 \quad = 2mr^2$

and l = distance of point of suspension from centre of mass = r

$$\therefore \qquad T = 2\pi\sqrt{\frac{2mr^2}{mgr}} = 2\pi\sqrt{\frac{2r}{g}}; \qquad \therefore \text{Angular frequency} \quad \omega = \frac{2\pi}{t} \quad \text{or} \quad \omega = \sqrt{\frac{g}{2r}}$$

Illustration 11: Find the period of small oscillations of a uniform rod with length l, pivoted at one end. **(JEE MAIN)**

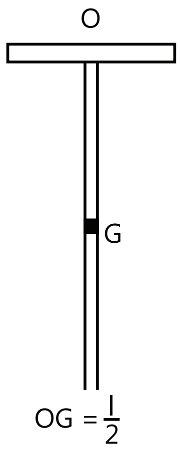


Figure 8.15

Sol: This is an example of a physical pendulum. Find moment of inertia about point of suspension and the distance of the point of suspension from the center of gravity.

$$T = 2\pi\sqrt{\frac{I_\circ}{mg(OG)}} \qquad \text{Here, } I_\circ = \frac{1}{3}ml^2 \quad \text{and } OG = \frac{l}{2}$$

$$\therefore \qquad T = 2\pi\sqrt{\frac{\left(\frac{1}{3}ml^2\right)}{(m)(g)\left(\frac{l}{2}\right)}} \qquad \text{or } T = 2\pi\sqrt{\frac{2l}{3g}}$$

Illustration 12: A uniform disc of radius 5.0 cm and mass 200 g is fixed at its center to a metal wire, the other end of which is fixed with a clamp. The hanging disc is rotated about the wire through angle and is released. If the disc makes torsional oscillations with time period 0.20 s, find the torsional constant of the wire. **(JEE MAIN)**

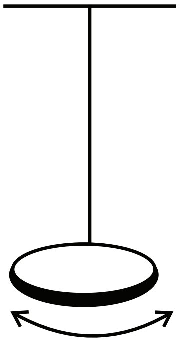


Figure 8.16

Sol: This is an example of a torsional pendulum. Find moment of inertia about the axis passing through the wire.

The Situation is shown in Fig 8.16. The moment of inertia of the disc about the wire is

$$I = \frac{mr^2}{2} = \frac{(0.200\text{kg})\left(5.0\times10^{-2}\text{m}\right)^2}{2} = 2.5\times10^{-4}\,\text{kg.m}^2$$

The time period is given by

$$T = 2\pi\sqrt{\frac{I}{K}}; \qquad K = \frac{4\pi^2 I}{T^2} = \frac{4\pi^2\left(2.5\times10^{-4}\,\text{kg}-\text{m}^2\right)}{(0.20\text{s})^2} = 0.25\frac{\text{kg}-\text{m}^2}{\text{s}^2}$$

10. SPRING - MASS SYSTEM

As shown in the Fig. 8.17 a mass m is attached to a massless spring. It is displaced from its mean position to a distance x. The restoring force is given by

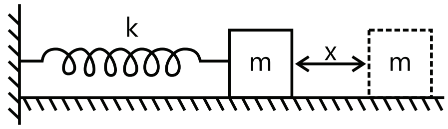


Figure 8.17: Block of mass m attached to spring

$F = -kx$ where k is the force constant.

$\Rightarrow ma = -kx;\quad a = -x\dfrac{k}{m}$ ...(i)

$\Rightarrow a \propto -x,\ \therefore$ Motion is SHM

$\Rightarrow \omega^2 = \dfrac{k}{m}$ or $\omega = \sqrt{\dfrac{k}{m}}$; $T = 2\pi\sqrt{\dfrac{m}{k}}$

10.1 Series and Parallel Combination of Springs

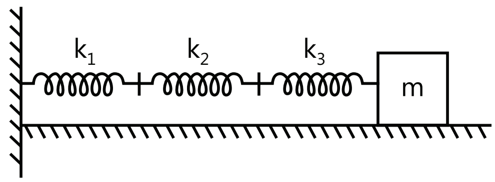


Figure 8.18: Series combination of springs

10.1.1. Serial Combination of Springs

If springs are connected in series, having force constants k_1, k_2, k_3 then the equivalent force

constant is $\dfrac{1}{k_{eff}} = \dfrac{1}{k_1} + \dfrac{1}{k_2} + \dfrac{1}{k_3}$

10.1.2 Parallel Combination of Springs

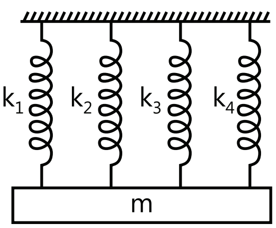


Figure 8.19: Parallel combination of springs

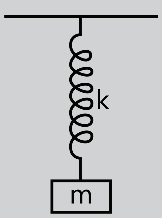
If springs are connected in parallel, then the effective force constant is given by

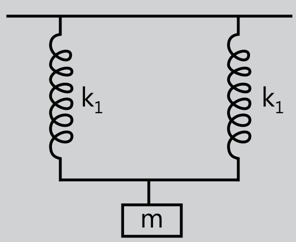
$k_{eff} = k_1 + k_2 + k_3 + \ldots\ldots\ldots\ldots\ldots$

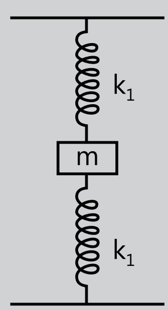
The force constant of a spring is inversely proportional to its length. If a spring of spring constant k is cut into two equal parts, the spring constant of each part becomes 2k. In general, if a spring of spring constant k is divided into n equal parts, the spring constant of each part is nk.

MASTERJEE CONCEPTS

Time Period for various scenarios at glance

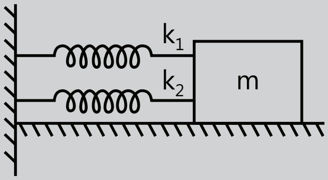


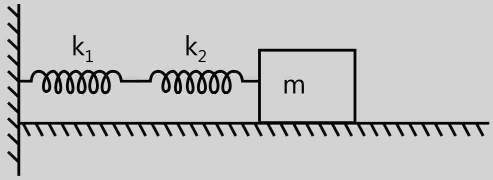




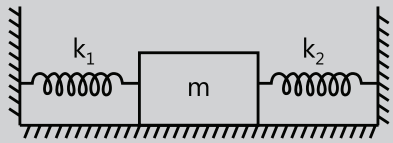
$T = 2\pi\sqrt{\dfrac{m}{k}}$ $\quad T = 2\pi\sqrt{\dfrac{m}{k_1 + k_2}}$ $\quad T = 2\pi\sqrt{\dfrac{m}{k_1 + k_2}}$

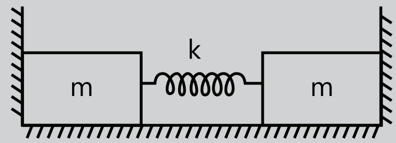
MASTERJEE CONCEPTS



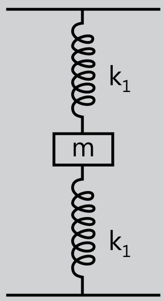


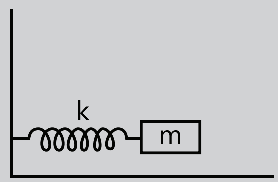
$$T = 2\pi\sqrt{\frac{m}{k_1 + k_2}} \qquad T = 2\pi\sqrt{\frac{m_1 m_2}{k(m_1 + m_2)}}$$





$$T = 2\pi\sqrt{\frac{m}{(k_1 + k_2)}} \qquad T = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$





$$T = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}} \qquad T = 2\pi\sqrt{\frac{m}{k}}$$

Figure 8.20

GV Abhinav (JEE 2012, AIR 329)

Illustration 13: For the arrangement shown in Fig 8.21, the spring is initially compressed by 3 cm. When the spring is released the block collides with the wall and rebounds to compress the spring again. **(JEE ADVANCED)**

(a) If the coefficient of restitution is $\frac{1}{\sqrt{2}}$, find the maximum compression in the spring after collision.

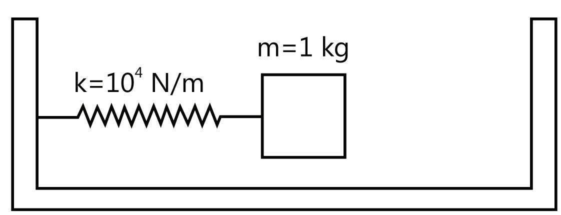


Figure 8.21

Sol: Conserve energy to find the velocity of the block. Use equation of restitution for collision of block with the wall.

(a) Velocity of the block just before

collision, $\frac{1}{2}mv_0^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_0^2$ or $v_0 = \sqrt{\frac{k}{m}\left(x_0^2 - x^2\right)}$

Here, $x_0 = 0.03\text{m}$, $x = 0.01\text{m}$, $k = 10^4 \text{N/m}$, $m = 1\text{kg}$ $\therefore$ $v_0 = 2\sqrt{2}\,\text{m/s}$

After collision, $v = ev_0 = \frac{1}{\sqrt{2}} 2\sqrt{2} = 2\,\text{m/s}$

Maximum compression in the spring

$$\frac{1}{2}kx_m^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \quad \text{or} \quad x_m = \sqrt{x^2 + \frac{m}{\kappa}v^2} = \sqrt{(0.01)^2 + \frac{1(2)^2}{10^4}}\,\text{m} \quad = 2.23\text{cm}$$

Illustration 14: Figure 8.22 shows a system consisting of a massless pulley, a spring of force constant k and a block of mass m. If the block is slightly displaced vertically down from its equilibrium position and released, find the period of its vertical oscillation in case (a), (b) and (c). **(JEE ADVANCED)**

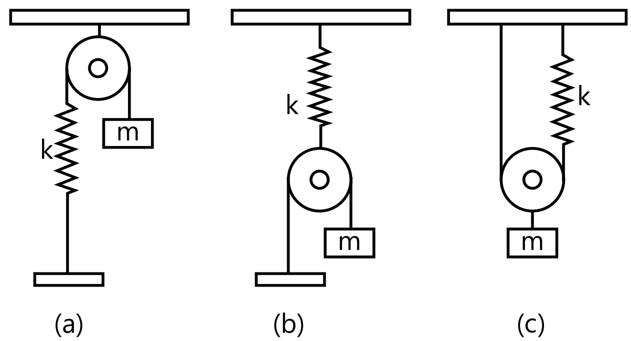


Figure 8.22

Sol: The restoring force on the block will depend on the elongation of the spring. For a small displacement of block find the elongation in the spring.

(a) In equilibrium, $kx_0 = mg$ ...(i)

When further depressed by an amount x, net restoring force (upwards) is,

$$F = -\{k(x + x_0) - mg\} = -kx \qquad (\text{as } kx_0 = mg)$$

$$a = -\frac{k}{m}x \quad \therefore \quad T = 2\pi\sqrt{\left|\frac{x}{a}\right|} \quad \text{or} \quad T = 2\pi\sqrt{\frac{m}{k}}$$

(b) In this case if the mass m moves down a distance x from its equilibrium position, then pulley will move down by $\frac{x}{2}$. So, the extra force in spring will be $k\frac{x}{2}$. Now, as the pulley is massless, this force $\frac{kx}{2}$ is equal to extra 2 T or T = $\frac{kx}{4}$. This is also the restoring force of the mass. Hence,

$$F = -\frac{kx}{4}; \quad a = -\frac{k}{4m}x \quad \text{or} \quad T = 2\pi\sqrt{\left|\frac{x}{a}\right|} \quad \text{or} \quad T = 2\pi\sqrt{\frac{4m}{k}}$$

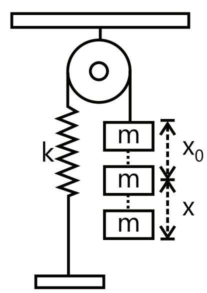


Figure 8.23

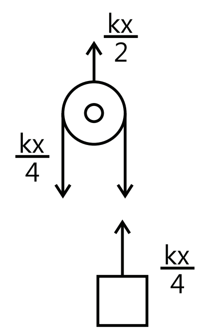


Figure 8.24

(c) In this situation if the mass m moves down a distance x from its equilibrium position, the pulley will also move by x and so the spring will stretch by 2x. Therefore, the spring force will be 2kx. The restoring force on the block will be 4kx.

$$\text{Hence,} \quad F = -4kx \quad \text{or} \quad a = -\frac{4k}{m},x$$

$$\therefore \quad T = 2\pi\sqrt{\left|\frac{x}{a}\right|} \quad \text{or} \quad T = 2\pi\sqrt{\frac{m}{4k}}$$

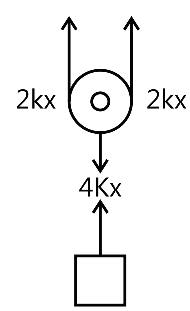


Figure 8.25

Illustration 15: A Spring mass system is hanging from the ceiling of an elevator in equilibrium. The elevator suddenly starts accelerating upwards with acceleration 'a' Find: (a) The frequency and (b) The amplitude of the resulting SHM. **(JEE MAIN)**

Sol: The time period of spring mass system does not depend on g or acceleration of elevator.

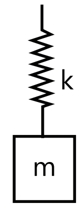


Figure 8.26

(a) Frequency = $2\pi\sqrt{\frac{m}{k}}$ (Frequency is independent of g in spring)

(b) Extension in spring in equilibrium in initial $= \frac{mg}{k}$

Extension in spring in equilibrium in accelerating lift $= \frac{m(g+a)}{k}$

$\therefore$ Amplitude $= \frac{m(g+a)}{k} - \frac{mg}{k} = \frac{ma}{k}$

11. BODY DROPPED IN A TUNNEL ALONG EARTH DIAMETER

Assume earth to be a sphere of radius R and center O. Let a tunnel be dug along the diameter of the earth as shown in Fig. 8.27. If a body of mass m is dropped at one end of the tunnel, the body executes SHM about the center of the earth. Let, at any instant body in the tunnel is at a distance y from the center O of the earth. Only the inner sphere of radius y will exert gravitational force F on the body as the body is inside the earth. The force F serves as the restoring force that tends to bring the body to the equilibrium position O.

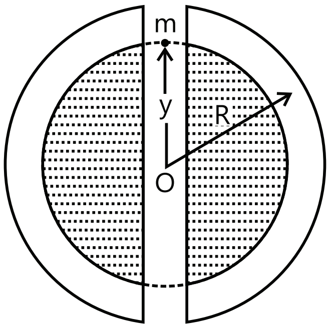


Figure 8.27: Body moving along diameter of earth

$\therefore$ Restoring force, $F = -G\frac{\left(4/3\,\pi y^3 \rho\right)m}{y^2}$

Where ρ is the density of the earth. The negative sign is assigned because the force is of attraction.

Acceleration of the body, $a = \frac{F}{m} = -\left(\frac{4}{3}\pi G\rho\right)y$ ... (i)

Now the quantity $(4/3)\pi G\rho$ is constant so that: $a \propto -y$

Thus the acceleration of the body is directly proportional to the displacement y and its direction is opposite to the displacement. Therefore, the motion of the body is simple harmonic.

$\therefore$ Time period, $T = 2\pi\sqrt{\frac{3}{4\pi G\rho}} = \sqrt{\frac{3\pi}{G\rho}}$ or $T = \sqrt{\frac{3\pi}{G\rho}}$ ... (ii)

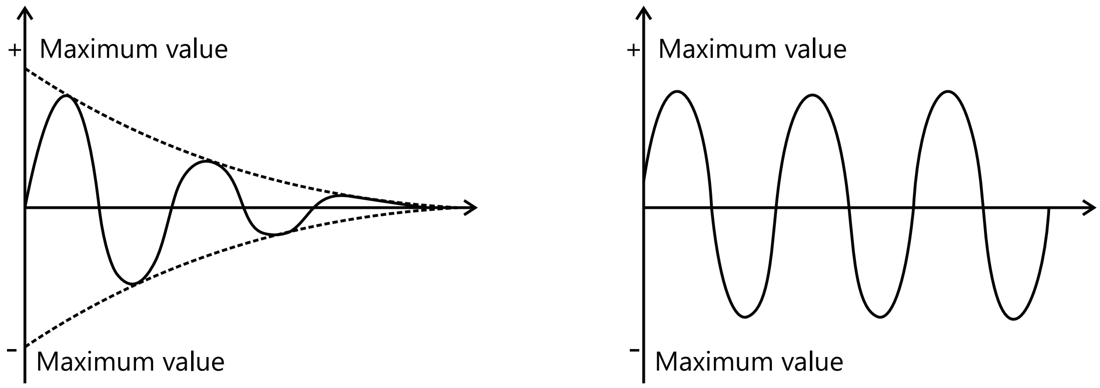
12. DAMPED AND UNDAMPED OSCILLATIONS

Damped oscillations is shown in the Fig 8.28 (a) given below. In such a case, during each oscillation, some energy is lost. The amplitude of the oscillation will be reduced to zero as no compensating arrangement for the less is provided. The only parameters that will remain unchanged are the frequency or time period. They will change only according to the circuit parameters.

As shown in Fig 8.28 (b), undamped oscillations have constant have amplitude oscillations.

Damping Force, $F_d = -bV = -b\frac{dx}{dt}$ where b is a constant giving the strength of damping. We can write Newton's law, now including damping force along with the restoring force. For a spring-mass system, we have,

$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$ or $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$; $x = ae^{-bt/2m}\cos(\omega t + \phi)$ ... (i) E.q. (i) describes sinusoidal motion whose amplitude (a) decreases exponentially with time. How fast the amplitude drops depends on the damping constants b and m. The frequency of this damped motion is given by: $f = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$



(a) Damped oscillation

(b) Undamped or Sustained osclliation

Figure 8.28: Damped and undamped oscillation

If the frictional forces are absent, b=0 so that: $f=\frac{1}{2\pi}\sqrt{\frac{k}{m}}$ (undamped oscillations)

13. FREE, FORCED AND RESONANT OSCILLATIONS

(a) **Free oscillations** are executed by an oscillating body that vibrates with its own frequency.

For example, when a simple pendulum is displaced from its mean position and then left free, it executes free oscillations. The natural frequency of the simple pendulum depends upon its length and is given by; $f=\frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$

(b) **Forced oscillations** - When a body is maintained in a state of oscillations by an external periodic force of frequency other than the natural frequency of the body, it executes forced oscillations.

The frequency of forced oscillations is equal to the frequency of the periodic force. The external applied force on the body is called the driver and the body set into oscillations is called driven oscillator.

Examples. (a) When the stem of a vibrating tuning fork is held in hand, only a feeble sound is heard. However, if the stem is pressed against a table top, the sound becomes louder. It is because the tuning fork forces the table to vibrate with fork's frequency. Since the table has a large vibrating area than the tuning fork, these forced oscillations produce a more intense sound.

Fig 8.29 shown the graph of forced oscillations as a function of ω.

At $\omega=\omega_a$, the value of A_0 is $\left(\frac{f_0}{b\omega}\right)$

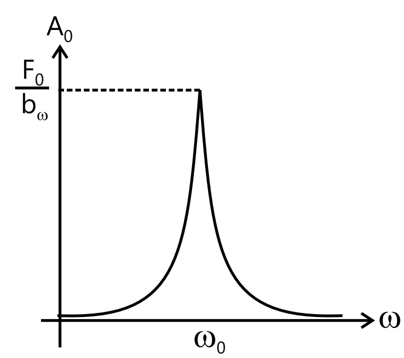


Figure 8.29: Forced oscillation

MASTERJEE CONCEPTS

Notice that amplitude of motion A_0 is directly proportional to the amplitude of driving force.

GV Abhinav (JEE 2012, AIR 329)

Mathematical analysis: Most of the oscillations that occur in systems (e.g. machinery) are forced oscillations; oscillations that are produced and sustained by an external force. The simplest driving force is one that oscillates as a sine or a cosine. Suppose such an external force F_{ext} is applied to an oscillator that moves along x axis such as a block connected to a spring. We can represent the external forces as: $F_{ext} = F_0 \cos\omega t$ Where F_0 is the maximum magnitude of the force and $\omega(= 2\pi f)$ is the angular frequency of the force. Then the equation of motion (with damping) is $ma = -kx - bV + F_0 \cos\omega t$. This equation can be written as

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_0 \cos\omega t \quad \text{or} \quad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos\omega t \qquad \text{... (i)}$$

The solution of eq. (i) is $x = A_0 \cos(\omega t + \phi)$ Where $A_0 = \dfrac{F_0/m}{\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(\dfrac{b\omega}{m}\right)^2}}$ ... (ii)

and $\omega_0 = \sqrt{k/m}$ is the frequency of undamped (b=0) oscillator i.e., natural frequency.

(iii) Resonant oscillations: When a body is maintained in a state of oscillations by a periodic force having the same frequency as the natural frequency of the body, the oscillations are called resonant oscillations. The phenomenon of producing resonant oscillations is called resonance.

(b) The amplitude of motion (A_0) depends on the difference between the applied frequency (ω) and natural frequency (ω_0). The amplitude is the maximum when the frequency of the driving force equals the natural frequency i.e., when $\omega = \omega_0$. It is because the denominator in eq. (ii) is the minimum when $\omega = \omega_0$. This condition is called resonance. When the frequency of the driving force equals ω_0, the oscillator is said to be in resonance with the driving force.

$$A_0 = \frac{F_0/m}{\sqrt{\left(\omega^2 - \omega_0^2\right) + \left(\dfrac{b\omega}{m}\right)^2}}$$

At resonance, $\omega = \omega_0$ and $A_0 = \dfrac{F_0/m}{\sqrt{(b\omega/m)^2}} = \dfrac{F_0}{b\omega}$

PROBLEM-SOLVING TACTICS

To verify SHM see whether force is directly proportional to y or see if $\dfrac{d^2x}{dt^2} + \omega^2 x = 0$ in cases when the equation is directly given compare with general equation to find the time period and other required answers

FORMULAE SHEET

1. Simple Harmonic Motion (SHM):

$F = -kx^n$

n is even - Motion of particle is not oscillatory

n is odd - Motion of particle is oscillatory.

If n = 1, F = -kx or $F \propto -x$. The motion is simple harmonic.

x = 0 is called the mean position or the equilibrium position.

Condition for SHM $\dfrac{d^2x}{dt^2} \propto -x$

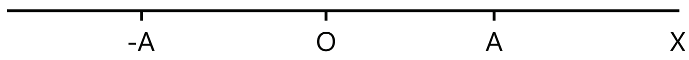


Figure 8.30

Acceleration, $a=\frac{F}{m}=-\frac{k}{m}x=-\omega^2 x$

Displacement $x=A\cos\underbrace{(\omega t+\phi)}_{\text{phase angle}}$ (A is Amplitude)

Time period of SHM $T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{m}{k}}$

Frequency ν of SHM $\nu=\frac{1}{T}=\frac{\omega}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Velocity of particle $v=\frac{dx}{dt}=-\omega A\sin(\omega t+\phi)$

Acceleration of particle $a=\frac{d^2x}{dt^2}=-\omega^2 A\cos(\omega t+\phi)=-\omega^2 x$

2. Energy in SHM:

Kinetic energy of particle $=\frac{1}{2}m\omega^2\left(A^2-x^2\right)=\frac{1}{2}k\left(A^2-x^2\right)$

Potential energy $U=\frac{1}{2}kx^2=\frac{1}{2}m\omega^2x^2$

Total energy E= P.E + K.E = $\frac{1}{2}m\omega^2A^2=\frac{1}{2}kA^2$

E is constant throughout the SHM.

3. Simple pendulum: Time period $T=2\pi\sqrt{\frac{\ell}{g_{eff}}}$

Here, ℓ is length of simple pendulum and $\vec{g}_{eff}=\vec{g}-\vec{a}$ where $\vec{g}$ is acceleration due to gravity and $\vec{a}$ is acceleration of the box or cabin etc. containing the simple pendulum.

4. Spring-block system: Time period $T=2\pi\sqrt{\frac{m}{k}}$

5. Physical pendulum: Time period $T=2\pi\sqrt{\frac{I}{mg\ell}}$

Here I is the moment of inertia about axis of rotation and ℓ is the distance of center of gravity from the point of suspension.

6. Torsional Pendulum:

$$T=2\pi\sqrt{\frac{I}{k}}$$

I is the moment of Inertia about axis passing through wire, k is torsional constant of wire.

7. Springs in series and parallel

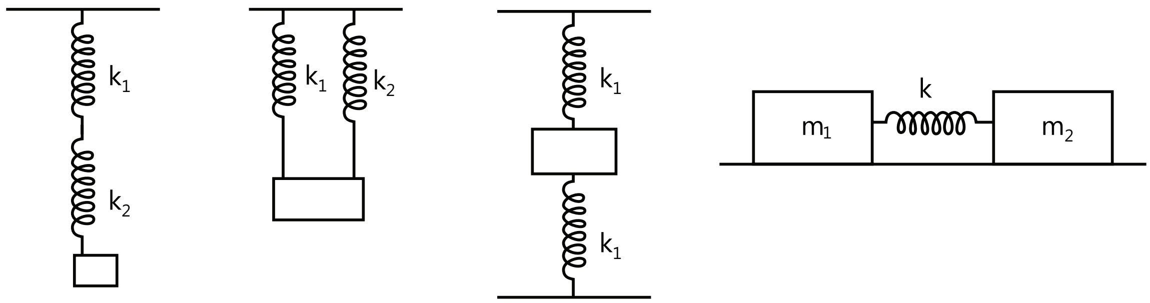


Figure 8.31

Series combination $\frac{1}{k}=\frac{1}{k_1}+\frac{1}{k_2}$

Parallel combination $k = k_1 + k_2$

8. For two blocks of masses m_1 and m_2 connected by a spring of constant k:

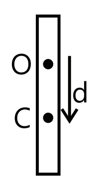
Time period $T=2\pi\sqrt{\frac{\mu}{k}}$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is reduced mass of the two-block system.

Solved Examples

JEE Main/Boards

Example 1: What is the period of pendulum formed by pivoting a meter stick so that it is free to rotate about a horizontal axis passing through 75 cm mark?



Sol: This is an example of a physical pendulum. Find moment of inertia about point of suspension and the distance of the point of suspension from the center of gravity.

Let m be the mass and ℓ be the length of the stick. $\ell = 100\text{cm}$ The distance of the point of suspension from center of gravity is $d = 25\,\text{cm}$

Moment of inertia about a horizontal axis through O is

$$I = I_c + md^2 = \frac{m\ell^2}{12} + md^2$$

$$T = 2\pi\sqrt{\frac{I}{mgd}}; \quad T = 2\pi\sqrt{\frac{\frac{m\ell^2}{12}+md^2}{mgd}}$$

$$T = 2\pi\sqrt{\frac{\ell^2+12d^2}{12gd}} = 2\pi\sqrt{\frac{\ell^2+12(0.25)^2}{12\text{x}9.8\text{x}0.25}} = 153\text{ s.}$$

Example 2: A particle executes SHM.

(a) What fraction of total energy is kinetic and what fraction is potential when displacement is one half of the amplitude?

(b) At what value of displacement are the kinetic and potential energies equal?

Sol: The sum of kinetic energy and potential energy is the total mechanical energy which is constant throughout the SHM.

We know that $E_{total} = \frac{1}{2}m\omega^2 A^2$

$KE = \frac{1}{2}m\omega^2\left(A^2 - X^2\right)$ and $U = \frac{1}{2}m\omega^2 x^2$

(a) When $x = \frac{A}{2}$, $KE = \frac{1}{2}m\omega^2\frac{3A^2}{4} \Rightarrow \frac{KE}{E_{total}} = \frac{3}{4}$