7. ROTATIONAL MECHANICS

1. INTRODUCTION

In this chapter we will be studying the kinematics and dynamics of a solid body in two kinds of motion. The first kind of motion of a solid body is rotation about a stationary axis, also called pure rotation. The second kind of motion of a solid is the plane motion wherein the center of mass of the solid body moves in a certain stationary plane while the angular velocity of the body remains permanently perpendicular to that plane. Here the body executes pure rotation about an axis passing through the center of mass and the center of mass itself translates in a stationary plane in the given reference frame. The axis through the center of mass is always perpendicular to the stationary plane. We will also learn about the inertia property in rotational motion, and the quantities torque and angular momentum which are rotational analogue of force and linear momentum respectively. The law of conservation of angular momentum is an important tool in the study of motion of solid bodies.

2. BASIC CONCEPT OF A RIGID BODY

A solid is considered to have structural rigidity and resists change in shape, size and density. A rigid body is a solid body which has no deformation, i.e. the shape and size of the body remains constant during its motion and interaction with other bodies. This means that the separation between any two points of a rigid body remains constant in time regardless of the kind of motion it executes and the forces exerted on it by surrounding bodies or a field of force.

A metal cylinder rolling on a surface is an example of a rigid body as shown in Fig. 7.1.

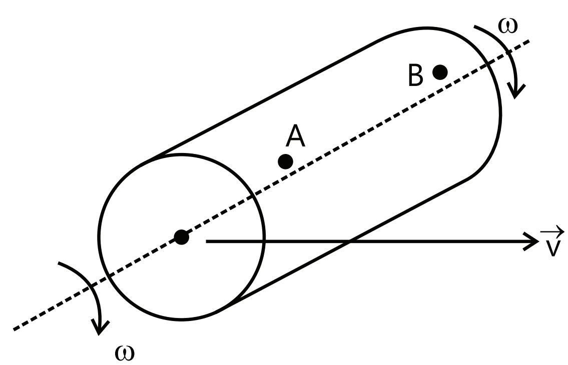


Figure 7.1: Metal cylinder rolling on a surface is a rigid body system. Relative distance between points A and B do not change.

Let velocities of points P and Q of a rigid body with respect to a reference frame be V_P and V_Q as shown in the Fig. 7.2.

As the body is rigid, the length PQ should not change during the motion of the body, i.e. the relative velocity between P and Q along the line joining P and Q should be zero i.e. velocity of approach or separation is zero. Let x-axis be along PQ, then

$\vec{V}_{QP}$ = relative velocity of Q with respect to P

$$\vec{V}_{QP} = (V_Q\cos\theta_2\,\hat{i} + V_Q\sin\theta_2\,\hat{j}) - (V_P\cos\theta_1\,\hat{i} - V_P\sin\theta_1\,\hat{j})$$

$$\vec{V}_{QP} = (V_Q\cos\theta_2 - V_P\cos\theta_1)\,\hat{i} + (V_P\sin\theta_1 + V_Q\sin\theta_2)\,\hat{j}$$

Now $V_P\cos\theta_1 = V_Q\cos\theta_2$

(Since velocity of separation is 0)

$\vec{V}_{QP} = (V_P\sin\theta_1 + V_Q\sin\theta_2)\hat{j}$ (which is perpendicular to line PQ).

Hence, we can conclude that for each and every pair of particles in a rigid body, relative motion between the two points in the pair will be perpendicular to the line joining the two points.

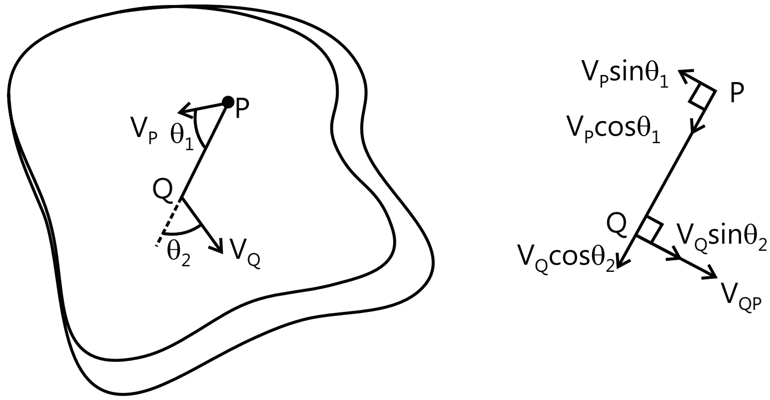


Figure 7.2: Relative velocity between two points of a rigid body

MASTERJEE CONCEPTS

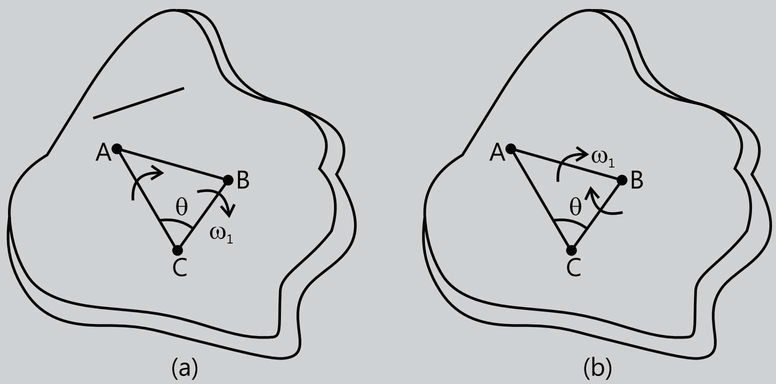


Figure 7.3: (a) Angular velocity of A and B w.r.t. C is ω_1 (b) Angular velocity of A and C w.r.t. B is ω_1

Suppose A, B, C are points of a rigid system hence during any motion the lengths of sides AB, BC, and CA will not change, and thus the angle between them will not change, and so they all must rotate through the same angle. Hence all the sides rotate by the same rate. Or we can say that each point is having the same angular velocity with respect to any other point on the rigid body.

Neeraj Toshniwal (JEE 2009 AIR 21)

3. MOTION OF A RIGID BODY

We will study the dynamics of three kinds of motion of a rigid body.

(a) Pure Translational motion

(b) Pure Rotational Motion

(c) Combined Translational and Rotational motion

Let us briefly discuss the characteristics of these three types of motion of a rigid body.

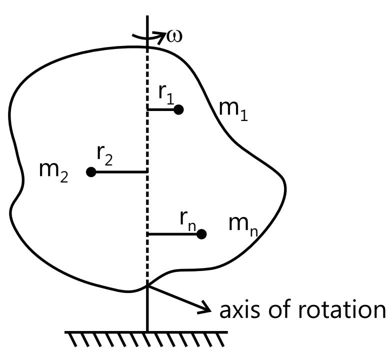


Figure 7.4: Body in pure rotational motion.

3.1 Pure Translational motion

A rigid body is said to be in pure translational motion if any straight line fixed to it remains parallel to its initial orientation all the time. E.g. a car moving along a straight horizontal stretch of a road. In this kind of motion, the displacement of each and every particle of the rigid body is the same during any time interval. All the points of the rigid body have the same velocity and acceleration at any instant. Thus to study the translational motion of a rigid body, it is enough to study the motion of an individual point belonging to that rigid body i.e. the dynamics of a point.

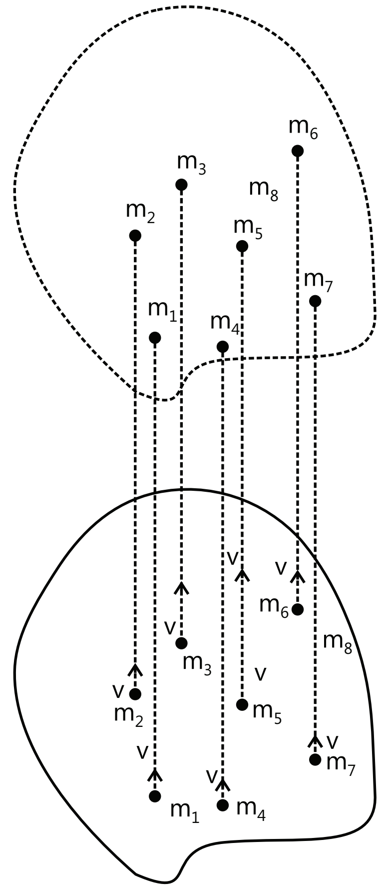


Figure 7.5: Body in pure translational motion.

3.2 Pure Rotational Motion

Suppose a rigid body of any arbitrary shape rotates about an axis which is stationary in a given reference frame. In this kind of motion every point of the body moves in a circle whose center lies on the axis of rotation at the foot of the perpendicular from the particle to this axis, and radius of the circle is equal to the perpendicular distance of the point from this axis. Every point of the rigid body moves through the same angle during a particular time interval. Such a motion is called pure rotational motion. Each particle has same instantaneous angular velocity (since the body is rigid) and different particles move in circles of different radii, the planes of all these circles are parallel to each other. Particles moving in smaller circles have less linear velocity and those moving in bigger circles have large linear velocity at the same instant.

In the Fig. 7.4 particles of mass m_1, m_2, m_3..... have linear velocities v_1, v_2, v_3....

If ω is the instantaneous angular velocity of the rigid body, then

$$v_1 = \omega r_1,\ v_2 = \omega r_2,\ v_3 = \omega r_3\ \ldots\ldots,\ v_n = \omega r_n$$

3.3 Combined Translational and Rotational Motion

A rigid body is said to be in combined translational and rotational motion if the body performs pure rotation about an axis and at the same time the axis translates with respect to a reference frame. In other words there is a reference frame K' which is rigidly fixed to the axis of rotation, such that the body performs pure rotation in the K' frame. The K' frame in turn is in pure translational motion with respect to a reference frame K. So to describe the motion of the rigid body in the K frame, the translational motion of K' frame is super-imposed on the pure rotational motion of the body in the K' frame.

Illustration 1: A body is moving down into a well through a rope passing over a fixed pulley of radius 10 cm. Assume that there is no slipping between rope and pulley. Calculate the angular velocity and angular acceleration of the pulley at an instant when the body is going down at a speed of 20 cm s^{-1} and has an acceleration of 4.0 m s^{-2}. **(JEE MAIN)**

Sol: Since the rope does not slip on the pulley, the linear speed and linear acceleration of the rim of the pulley will be equal to the speed and acceleration of the body respectively.

Therefore, the angular velocity of the pulley is

$$\omega = \frac{\text{linear velocity of rim}}{\text{radius of rim}} = \frac{20\text{ cm s}^{-1}}{10\text{ cm}} = 2\text{ rad s}^{-1}$$

And the angular acceleration of the pulley is

$$\alpha = \frac{\text{linear acceleration of rim}}{\text{radius of rim}} = \frac{4.0\text{ ms}^{-2}}{10\text{cm}} = 40\text{ rad s}^{-2}$$

4. ROTATIONAL KINEMATICS

Suppose a rigid body performing pure rotational motion about an axis of rotation rotates by an angle Δθ in a time interval Δt. The instantaneous angular velocity ω, is defined as,

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad \text{...(i)}$$

Similarly, the instantaneous angular acceleration α is defined as,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \qquad \text{...(ii)}$$

The relations between linear distance s, linear velocity v and linear acceleration a, and the corresponding angular variables describing circular motion θ, ω, and α respectively are given as:

$$s = r\theta; \quad v = r\omega; \qquad a_t = r\alpha \qquad \text{...(iii)}$$

Here the subscript t along with a in the expression for acceleration signifies that this is the tangential component of linear acceleration.

If a body rotates with uniform angular acceleration,

$$\omega = \omega_0 + \alpha t \ ; \ \theta = \omega_0 t + \frac{1}{2}\alpha t^2 ; \ \omega^2 = \omega_0^2 + 2\alpha\theta \qquad \text{...(iv)}$$

where ω_0 is initial angular velocity.

The equations for angular displacement, angular velocity and angular acceleration are similar to the corresponding equations of linear motion.

Illustration 2: A disc starts rotating with constant angular acceleration of $\pi / 2 \text{ rad s}^{-2}$ about a fixed axis perpendicular to its plane and through its center. Calculate

(a) The angular velocity of the disc after 4 s

(b) The angular displacement of the disc after 4s and

(c) Number of turns accomplished by the disc in 4 s. **(JEE MAIN)**

Sol: Use the first and second equations of angular motion with constant angular acceleration.

Here $\alpha = \frac{\pi}{2} \text{ rad s}^{-2}$; $\omega_0 = 0$; $t = 4$ s;

(a) $\omega(4 \text{ s}) = 0 + \left(\frac{\pi}{2}\text{rad s}^{-2}\right) \times 4 \text{ s} = 2\pi \text{ rad s}^{-1}$

(b) $\theta_{(4s)} = 0 + \frac{1}{2}\left(\frac{\pi}{2}\text{rad s}^{2}\right) \times (16\text{s}^2) = 4\pi \text{ rad}$

(c) $\Rightarrow n \times 2\pi \text{ rad} = 4\pi \text{ rad} \Rightarrow n = 2.$

MASTERJEE CONCEPTS

For variable angular acceleration we should proceed with differential equation $\frac{d\omega}{dt} = \alpha$

Akshat Kharaya (JEE 2009 AIR 235)

5. MOMENT OF INERTIA

Before discussing the dynamics of rigid body motion let us study about an important property of a rigid body called Moment of Inertia which is indispensable in understanding its dynamics.

Physical Significance of Moment of Inertia: As the name suggests, moment of inertia is the measure of the rotational inertia property of a rigid body, the rotational analog of mass in translational motion. "It is the property of the rigid body by virtue of which it opposes any change in its state of uniform rotational motion." The moment of inertia of a rigid body depends on its mass, on the location and orientation of the axis of rotation and on the shape and size of the body or in other words on the distribution of the mass of the body with respect to the axis of rotation. SI units of moment of inertia is Kg-m^2. Moment of inertia about a particular axis of rotation is a scalar positive quantity.

Definition: Moment of inertia of a system of n particles about an axis is defined as:

$$I = m_1r_1^2 + m_2r_2^2 + ...+ m_nr_n^2 \text{ i.e. } I = \sum_{i=1}^{n} m_i r_i^2 \qquad ...(i)$$

where, r_i is the perpendicular distance of ith particle of mass m_i from the axis of rotation.

For a continuous rigid body, the moment of inertia can be calculated as:

$$I = \int r^2(dm) \qquad ...(ii)$$

where dm is the mass of an infinitesimal element of the body at a perpendicular distance r from the axis of rotation.

Moment of inertia depends on:

(a) Mass of the rigid body.

(b) Shape and size of the rigid body.

(c) Location and orientation of the axis of rotation.

MASTERJEE CONCEPTS

Moment of inertia does not change if the mass:

(i) Is shifted parallel to the axis of rotation because r_i does not change.

(ii) Is rotated about the axis of rotation in a circular path because r_i does not change.

Chinmay S Purandare (JEE 2012 AIR 698)

Illustration 3: Two particles having masses m_1 & m_2 are situated in a plane perpendicular to line AB at a distance of r_1 and r_2 respectively as shown.

(i) Find the moment of inertia of the system about axis AB?

(ii) Find the moment of inertia of the system about an axis passing though m_1 and perpendicular to the line joining m_1 and m_2.

(iii) Find the moment of inertia of the system about an axis passing through m_1 and m_2.

(iv) Find moment of inertia about an axis passing though center of mass and perpendicular to line joining m_1 and m_2 . **(JEE MAIN)**

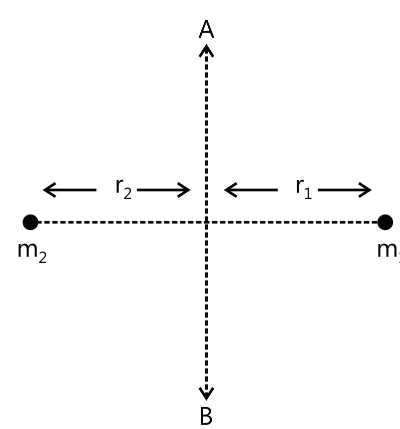


Figure 7.6

Sol: Use the formula for moment of inertia of a system of n particles. Find the distance of center of mass from m_1.

(i) Moment of inertia of particle on right is $I_1 = m_1r_1^2$

Moment of inertia of particle on left is $I_2 = m_2 r_2^2$

Moment of inertia of the system about AB is $I = I_1 + I_2 = m_1 r_1^2 + m_2 r_2^2$

(ii) Moment of inertia of particle on right is $I_1 = 0$

Moment of inertia of particle on left is $I_2 = m_2 (r_1 + r_2)^2$

Moment of inertia of the system about axis is $I = I_1 + I_2 = 0 + m_2 (r_1 + r_2)^2$

(iii) Moment of inertia of particle on right is $I_1 = 0$

Moment of inertia of particle of left is $I_2 = 0$

Moment of inertia of the system about axis is $I = I_1 + I_2 = 0 + 0$

(iv) of system $r_{cm} = m_2 \left(\frac{r_1 + r_2}{m_1 + m_2} \right)$ = Distance of center mass from mass m_1

Distance of center of mass from mass $m_2 = m_1 \left(\frac{r_1 + r_2}{m_1 + m_2} \right)$

So moment of inertia about center of mass = $I_{cm} = m_1 \left(m_2 \frac{r_1 + r_2}{m_1 + m_2} \right)^2 + m_2 \left(m_1 \frac{r_1 + r_2}{m_1 + m_2} \right)^2$

$$I_{cm} = \frac{m_1 m_2}{m_1 + m_2} (r_1 + r_2)^2 .$$

Illustration 4: Three particles each of mass m, are situated at the vertices of an equilateral triangle PQR of side a as shown in the Fig 7.7. Calculate the moment of inertia of the system about

(i) The line PX perpendicular to PQ in the plane of PQR.

(ii) One of the sides of the triangle PQR

(iii) About an axis passing through the centroid and perpendicular to plane of the triangle PQR. **(JEE MAIN)**

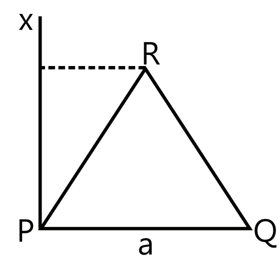


Figure 7.7

Sol: Use the formula for moment of inertia of a system of n particles.

(i) Perpendicular distance of P from PX = 0; perpendicular distance of Q from PX = a perpendicular distance of R from PX = a/2. Thus, the moment of inertia of the particle at P is 0, that of particle Q is ma^2, and of the particle at R is $m(a/2)^2$.

The moment of inertia of the three particle system about PX is $I = 0 + ma^2 + m(a/2)^2 = \frac{5ma^2}{4}$

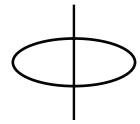
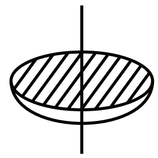
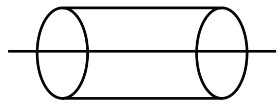
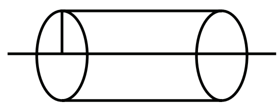
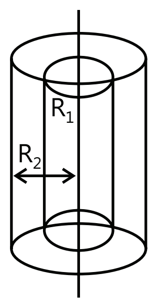
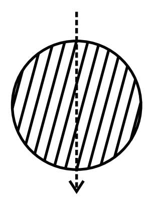
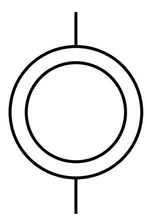
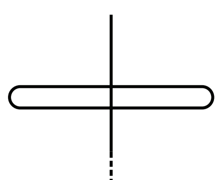
Note that the particles on the axis do not contribute to the moment of inertia.

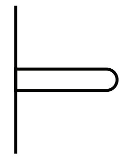
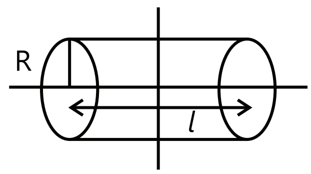
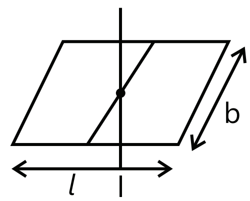
(ii) Moment of inertia about the side PR = mass of particle Q × square of perpendicular distance of Q from side PR,

$$I_{PR} = m \left(\frac{\sqrt{3}}{2} a \right)^2 = \frac{3ma^2}{4}$$

(iii) Distance of centroid from each of the particles is $\frac{a}{\sqrt{3}}$, so moment of inertia about an axis passing through the centroid and perpendicular to the plane of triangle PQR = $I_C = 3m \left(\frac{a}{\sqrt{3}} \right)^2 = ma^2$

Table 7.1: Formulae of MOI of symmetric bodies

S. No.	Body, mass M	Axis	Figure	I	K(Radius of Gyration)
1.	Ring or loop of radius R	Through its center and perpendicular to its plane		MR^2	R
2.	Disc, radius R	Perpendicular to its plane through its center		$\frac{MR^2}{2}$	$\frac{R}{\sqrt{2}}$
3.	Hollow cylinder, radius R	Axis of cylinder		MR^2	R
4.	Solid cylinder, radius R	Axis of cylinder		$\frac{MR^2}{2}$	$\frac{R}{\sqrt{2}}$
5.	Thick walled cylinder,	Axis of cylinder		$\frac{M\left(R_1^2+R_2^2\right)}{2}$	$\frac{\sqrt{R_1^2+R_2^2}}{2}$
6.	Solid sphere, radius R	Diameter		$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$
7.	Spherical shell radius, R	Diameter		$\frac{2MR^2}{3}$	$\sqrt{\frac{2}{3}}R$
8.	Thin rod, length L	Perpendicular to rod at middle point		$\frac{ML^2}{12}$	$\frac{L}{2\sqrt{3}}$

9.	Thin rod, length L	Perpendicular to rod at one end		$\frac{ML^2}{3}$	$\frac{L}{\sqrt{3}}$
10.	Solid cylinder, length l	Through center and perpendicular to length		$\frac{MR^2}{4}+\frac{Ml^2}{12}$	$\sqrt{\frac{R^2}{4}+\frac{l^2}{12}}$
11.	Rectangular sheet, length l and breadth b	Through center and perpendicular to plane		$\frac{M(l^2+b^2)}{12}$	$\sqrt{\frac{l^2+b^2}{12}}$

MASTERJEE CONCEPTS

While deriving the MOI *of* any rigid body the element chosen should be such that:

Either perpendicular distance of axis from each point of the element is same or the moment of inertia of the element about the axis of rotation is known.

Nitin Chandrol (JEE 2012 AIR 134)

5.1 Theorems on Moment of Inertia

1. Theorem of Parallel Axes: This theorem is very useful in cases when the moment of inertia about an axis z_C passing through the center of mass (C.O.M) of the rigid body is known, and we sought to find the moment of inertia about any other axis z which is parallel to the axis z_C as shown in Fig. 7.8. The moment of inertia of the rigid body about axis z is equal to the sum of the moment of inertia about axis z_C and the product of the mass m of the body by the square of perpendicular distance between the two axes. If the moment of inertia of the rigid body about axis z_C is I_C, then the moment of inertia I of this body about any parallel axis z, is given by $I = I_C + Md^2$ ...(i)

where d is the perpendicular distance between the two axes.

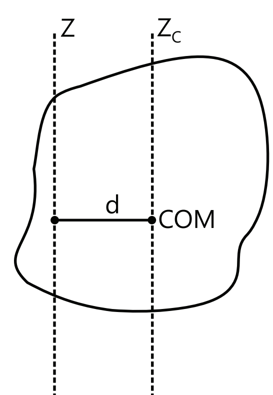


Figure 7.8: Parallel axes

Illustration 5: Find the moment of inertia of a uniform sphere of mass m and radius R about a tangent if the sphere is (i) solid (ii) hollow **(JEE MAIN)**

Sol: We know the formula for moment of inertia of sphere about an axis passing through its center. Use the parallel axes theorem to find the moment of inertia about the tangent.

(i) Using parallel axis theorem

$I = I_C + md^2$

For solid sphere

$I_C = \frac{2}{5}mR^2$, $d = R$; $\quad I = \frac{7}{5}mR^2$

(ii) Using parallel axis theorem

$I = I_C + md^2$

For hollow sphere

$I_C = \frac{2}{3}mR^2$, $d = R$; $I = \frac{5}{3}mR^2$

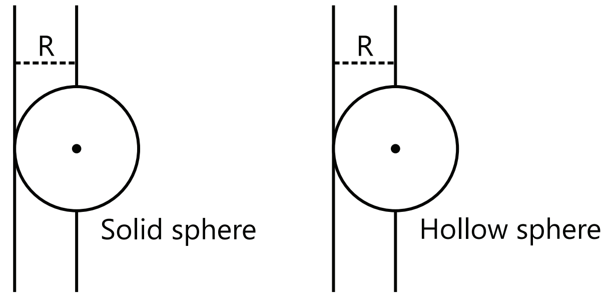


Figure 7.9

Illustration 6: Find the moment of inertia of the two uniform joint roads having mass m each about point P as shown in Fig 7.10. Use parallel axis theorem. **(JEE MAIN)**

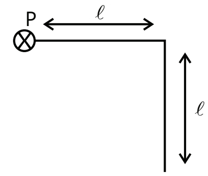


Figure 7.10

Sol: We know the formulae for moment of inertia of rod about the axes passing through its center and through one of its ends and perpendicular to it. Use the parallel axes theorem to find the moment of inertia about the point P.

Moment of inertia of rod 1 about axis P, $I_1 = \frac{m\ell^2}{3}$

Moment of inertia of rod 2 about axis p, $I_2 = \frac{m\ell^2}{12} + m\left(\sqrt{5}\frac{\ell}{2}\right)^2$

So moment of inertia of a system about axis p; $I = \frac{5m\ell^2}{3}$

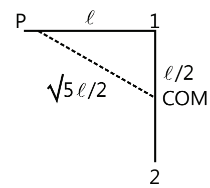


Figure 7.11

2. Theorem of Perpendicular Axes: This theorem is applicable only in case of two dimensional rigid body or planar lamina as shown in Fig. 7.12. Let the lamina lie in the x-y plane and I_x and I_y be the moment of inertia of the lamina about x and y axes respectively then the moment of inertia about z-axis perpendicular to the plane of the lamina and passing through the point of intersection of x and y axes is given as:

$$I_z = I_x + I_y \quad \text{...(ii)}$$

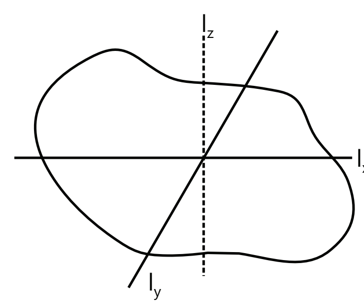


Figure 7.12: Perpendicular axes

Illustration 7: Find the moment of inertia of a half-disc about an axis perpendicular to the plane and passing through its center of mass. Mass of this disc is M and radius is R. **(JEE MAIN)**

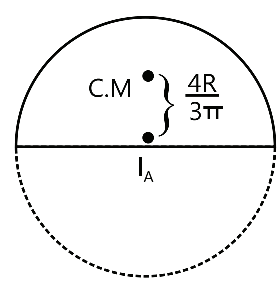


Figure 7.13

Sol: We know the formula for the moment of inertia of the half disc about a perpendicular axis through the center A. Use the parallel axes theorem to find the moment of inertia about a perpendicular axis through the center of mass.

The COM of half disc will be at distance $4R/3\pi$ from the center A. Let moment of inertia of half disc about a perpendicular axis passing through A be I_A.

First we fill the remaining half with same density to get a full disc of mass 2M.

The moment of inertia about center A of full disc will be $2I_A$,

So, $I_A = \frac{2MR^2}{2\times 2} = \frac{MR^2}{2}$; $I_A = I_{CM} + M \times \left(\frac{4R}{3\pi}\right)^2$; $I_{CM} = \frac{MR^2}{2} - M\times\left(\frac{4R}{3\pi}\right)^2$

Illustration 8: Calculate the moment of inertia of a uniform disc of mass M and radius R about a diameter.

(JEE MAIN)

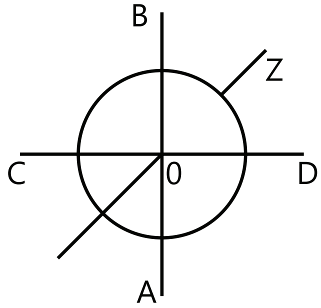


Figure 7.14

Sol: For a uniform disc all diameters are equivalent, i.e. moment of inertia about any diameter will be equal to that about any other diameter. We know the formula for moment of inertia of disc about axis perpendicular to its plane and passing through its center. Use the perpendicular axes theorem to find the moment of inertia about a diameter.

Let AB and CD be two mutually perpendicular diameters of the disc. Take them as x and y axes and the line perpendicular to the plane of the disc through the center as the Z – axis. The moment of inertia of the ring about the Z – axis is $I = \frac{1}{2}MR^2$. As the disc is uniform, all of its diameters are equivalent and so $I_x = I_y$

From perpendicular an axis theorem $I_z = I_x + I_y$; hence $I_x = \frac{I_z}{2} = \frac{MR^2}{4}$

Illustration 9: In the Fig 7.15 shown find the moment of inertia of square plate having mass m and sides a about axis 2 passing through point C (center of mass) and in the plane of plate. **(JEE MAIN)**

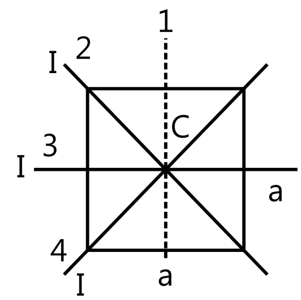


Figure 7.15

Sol: For uniform square plate axes 2 and 4 along diagonals are equivalent and axes 1 and 3 are equivalent. Suppose I_c is the moment of inertia about the axis perpendicular to the plane of plate and passing through the center C. Use perpendicular axes theorem to prove that the axes 1 and 2 are also equivalent.

Using perpendicular axes theorem $I_C = I_4 + I_2 = I' + I' = 2I'$ .... (i)

Using perpendicular axes theorem $I_C = I_3 + I_1 = I + I = 2I$ ....(ii)

From (i) and (ii) we get $I' = I$

$I_c = 2I = \frac{ma^2}{6} \Rightarrow I' = \frac{ma^2}{12}$

5.2 Radius of Gyration

The radius of gyration of a rigid body about an axis z is equal to the radius of a ring whose mass is equal to the mass of the rigid body, and the moment of inertia of the ring about an axis passing through its center and perpendicular to its plane is equal to the moment of inertia of the rigid body about the axis z. Radius of gyration can also be defined as the perpendicular distance from the axis of rotation where all mass of the rigid body can be assumed to be concentrated when the rigid body is performing pure rotation to get the equation of motion of the body. Thus, the radius of gyration is the ''equivalent distance'' of the rigid body from the axis of rotation.

$I = MK^2$

I = Moment of inertia of the rigid body about an axis

M = Mass of the rigid body

K = Radius of gyration about the same axis

or $K = \sqrt{\frac{I}{M}}$ ...(iii)

Length K is the property of the rigid body which depends upon the shape and size of the body and on the orientation and location of the axis of rotation. S.I. Unit of K is meter.

Illustration 10: Find the radius of gyration of a hollow uniform sphere of radius R about its tangent. **(JEE MAIN)**

Sol: Use the formula for radius of gyration.

Moment of inertia of a hollow sphere about a tangent = $\frac{5}{3}MR^2$

$$MK^2 = \frac{5}{3}MR^2 \Rightarrow K = \sqrt{\frac{5}{3}}R$$

5.3 Moment of Inertia of a Body Having a Cavity

If we know the moments of inertia of different parts of a rigid body about the same axis, then the moment of inertia of the entire body can be calculated by simply adding the moments of inertia of the different parts (about the same axis) i.e. moment of inertia is an additive quantity. This principle can be used to calculate the moment of inertia of a body having hollow spaces by first assuming the hollow spaces to be filled with same density as that of the body and evaluating the moment of inertia of the whole body about the given axis and then add the moments on inertia of the hollow spaces about the same axis considering them to have negative mass.

Illustration 11: A uniform disc of radius R has a round disc of radius R/3 cut as shown in Fig 7.16. The mass of the disc equals M. Find the moment of inertia of such a disc relative to the axis passing through geometrical center of original disc and perpendicular to the plane of the disc. **(JEE ADVANCED)**

Sol: Consider the whole disc without the cavity. The cavity can be thought of as a negative mass of same density as disc. We know the formula for moment of inertia of uniform disc about axis perpendicular to its plane and passing through its center. Find the moment of inertia of cavity (negative mass) about the perpendicular axis passing through center of whole disc. The moment of inertia of disc with cavity is the sum of the moment of inertia of whole disc and the moment of inertia of cavity (negative).

Let the mass per unit area of the material of disc be σ. Now the empty space can be considered as having density $-\sigma$

Now $I_0 = I_\sigma + I_{-\sigma}$

$$I_\sigma = (\sigma\pi R^2)R^2/2 = \text{MI of } \sigma \text{ about O} = MR^2/2$$

$$I_{-\sigma} = \frac{-\sigma\pi(R/3)^2(R/3)^2}{2} + [-\sigma\pi(R/3)^2](2R/3)^2$$

$$= \text{M.I of } -\sigma \text{ About O} = -MR^2/18$$

$$I_0 = MR^2/2 - MR^2/18$$

$$I_0 = \frac{4}{9}MR^2$$

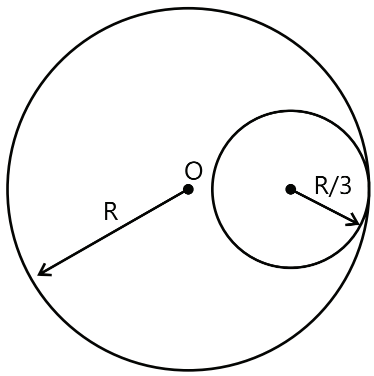


Figure 7.16

6 TORQUE

6.1 Torque About a Point

Torque of force F relative to a point O is defined as

$$\vec{\tau} = \vec{r} \times \vec{F} \quad ...(x)$$

where $\vec{F}$ = force applied to a point on a body

$\vec{r}$ = position vector of the point of application of force relative to the point O in a chosen reference frame about which we want to determine the torque (see Fig. 7.17).

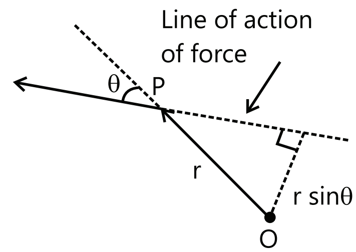


Figure 7.17: Torque of a force

Torque is a vector quantity and its direction is given by the right hand rule for cross product of vectors.

Magnitude of torque $|\vec{\tau}| = r F \sin\theta = r_{\perp}F = rF_{\perp}$

where θ is the angle between the force $\vec{F}$ and the position vector $\vec{r}$ of point of application.

$r_{\perp} = r \sin\theta$ = perpendicular distance of line of action of force from point O.

$F_{\perp} = F \sin\theta$ = component of $\vec{F}$ perpendicular to $\vec{r}$

SI unit of torque is N-m.

Illustration 12: Find the torque about point O and A. **(JEE MAIN)**

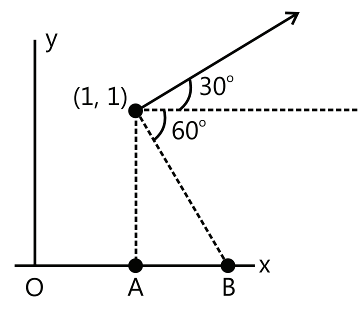


Figure 7.18

Sol: Express the position vector of A relative to O in terms of unit vectors $\hat{i}$ and $\hat{j}$. Force is given in terms of unit vectors $\hat{i}$ and $\hat{j}$.

Torque about point O, $\vec{\tau} = \vec{r_0} \times \vec{F}$, $\vec{r_0} = \hat{i} + \hat{j}$, $\vec{F} = 5\sqrt{3}\,\hat{i} + 5\hat{j}$

$\vec{\tau} = (\hat{i} + \hat{j}) \times (5\sqrt{3}\hat{i} + 5\hat{j}) = 5(1-\sqrt{3})\hat{k}$

Torque about point A, $\vec{\tau} = \vec{r_a} \times \vec{F}$, $\vec{r_a} = \hat{j}$, $\vec{F} = 5\sqrt{3}\,\hat{i} + 5\hat{j}$

$\vec{\tau} = \hat{j} \times (5\sqrt{3}\hat{i} + 5\hat{j}) = -5\sqrt{3}\hat{k}$

Illustration 13: A particle of mass m is released in vertical plane from a point on the x – axis, it falls vertically along the y – axis. Find the torque τ about origin? **(JEE MAIN)**

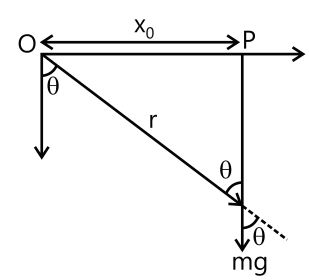


Figure 7.19

Sol: Torque is produced by the force of gravity. This will be equal to the product of force of gravity and the perpendicular distance between the line of action of force of gravity and the origin O.

$\vec{\tau} = rF \sin\theta\,\hat{k}$ Or $\tau = r_{\perp}F = x_0 mg$

$= r\,mg\dfrac{x_0}{r} = mgx_0\hat{k}$

6.2 Torque About An Axis

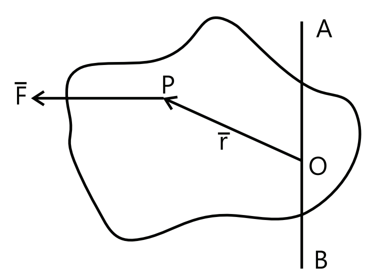


Figure 7.20: Torque about an axis

The torque of a force $\vec{F}$ about an axis AB is the component of the torque of $\vec{F}$ about point A along the axis AB.

Alternatively to find torque of force $\vec{F}$ about axis AB we choose any point O on the axis AB and find the torque of $\vec{F}$ about O as $\vec{\tau}_0 = \vec{r} \times \vec{F}$. Then we calculate the component of $\vec{\tau}_0$ along AB to get $\vec{\tau}_{AB}$ (see Fig. 7.20).

There are a few special cases of torque of a force about an axis:

Case I: Applied force is parallel to the axis of rotation, i.e. $\vec{F} \parallel \overrightarrow{AB}$

Therefore torque $\vec{r} \times \vec{F}$ about any point on the axis will be perpendicular to $\vec{F}$ and hence perpendicular to $\overrightarrow{AB}$. Therefore the component of $\vec{r} \times \vec{F}$ along $\overrightarrow{AB}$ will be zero.

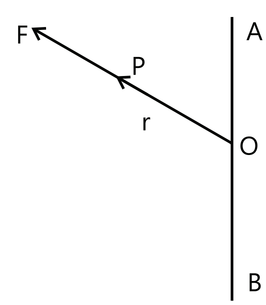


Figure 7.21: Force intersects axis

Case II: The line of action of the applied force intersects the axis of rotation ($\vec{F}$ intersects $\overrightarrow{AB}$)

If we choose the point of intersection of line of action of $\vec{F}$ and $\overrightarrow{AB}$ as the origin O then the position vector $\vec{r}$ and applied force $\vec{F}$ will be collinear (see Fig. 7.21). Therefore the torque about O is $\vec{r} \times \vec{F} = 0$ and thus the component of this torque along line AB will also be zero.

Case III: Line of action of $\vec{F}$ and axis AB are skew and $\vec{F} \perp \overrightarrow{AB}$

Let O be the origin on the axis AB and P be the point of application of force $\vec{F}$ such that OP is perpendicular to the axis AB (see Fig. 7.22).

Then torque $\vec{\tau} = \overrightarrow{OP} \times \vec{F}$ will be parallel to axis AB and the component of $\vec{\tau}$ along AB will be equal to its magnitude i.e.

$$\tau_{AB} = F\times(OP)\sin\theta = F\times l$$

where ℓ = (OP)sinθ is the length of the common perpendicular to the line of action of force and the axis called the lever arm or moment arm of this force.

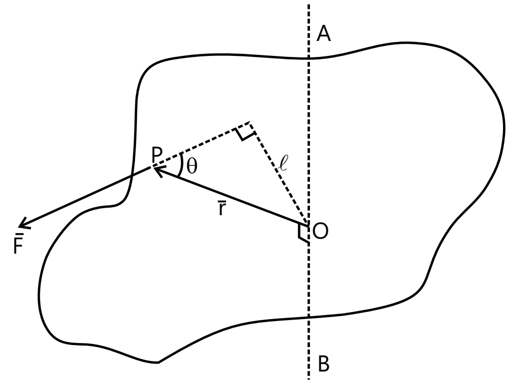


Figure 7.22: Force and axis are skew

Illustration 14: Find the torque of weight about the axis passing through point P. **(JEE MAIN)**

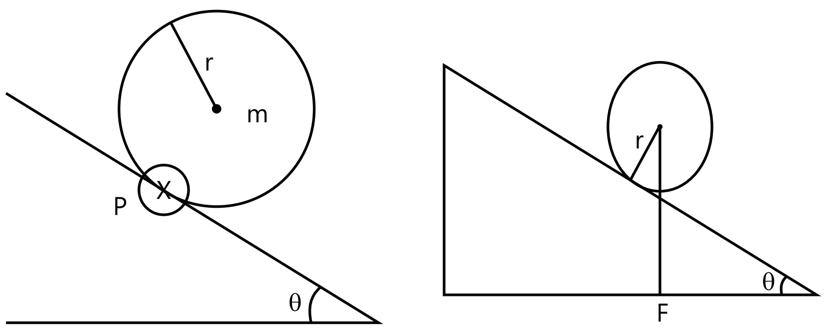


Figure 7.23

Sol: Required torque is equal to the product of force of gravity and the perpendicular distance between the line of action of force of gravity and the point P.

$\vec{F}$ = mg Downwards

$\vec{\tau} = \vec{F} \times \vec{r} \quad = F.r\sin\theta$

Illustration 15: A bob of mass m is suspended at point O by string of length ℓ. Bob is moving in a horizontal circle find out. (i) Torque of gravity and tension about point O and O′ (ii) Net torque about axis OO′. **(JEE ADVANCED)**

Sol: Torque of a force about an origin is equal to the product of force and the perpendicular distance between the line of action of force and the origin.

(i) Torque about point O

Torque of tension (T), τ_{net} = 0 (tension is passing through point O)

Torque of gravity $\tau_{mg} = mg\ell\sin\theta$ (along negative $\hat{j}$)

Torque about point O′

Torque of gravity $\tau_{mg} = mgr = mg\ell\sin\theta$ (along negative $\hat{j}$)

Torque of tension $\tau_T = Tr\sin(90+\theta)$ ($T\cos\theta = mg$)

$$\tau_T = Tr\cos\theta = \frac{mg}{\cos\theta}(\ell\sin\theta)\cos\theta = mg\ell\sin\theta \text{ (along positive } \hat{j})$$

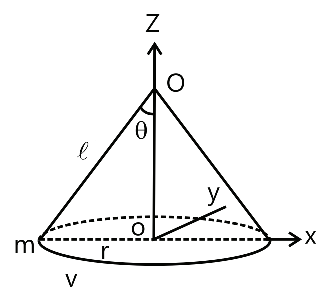


Figure 7.24

(ii) Torque about axis OO′

Torque of gravity about axis OO′ τ_{mg} = 0 (force mg is parallel to axis OO′)

Torque of tension about axis OO′ τ_T = 0 (force T intersects the axis OO′)

Net torque about axis OO′ τ_{net} = 0

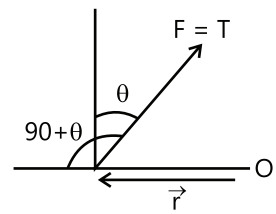


Figure 7.25

6.3 Force Couple

A pair of forces each of same magnitude and acting in opposite directions is called a force couple.

Torque due to couple = magnitude of one force x distance between their lines of action.

Magnitude of torque = τ = F (d)

A couple does not exert a net force on an object even though it exerts a torque.

Net torque due to a force couple is the same about any point (see Fig. 7.26).

Total torque about A = $x_1F + x_2F = F(x_1 + x_2) = Fd$

Total torque about B = $y_1F - y_2F = F(y_1 - y_2) = Fd$

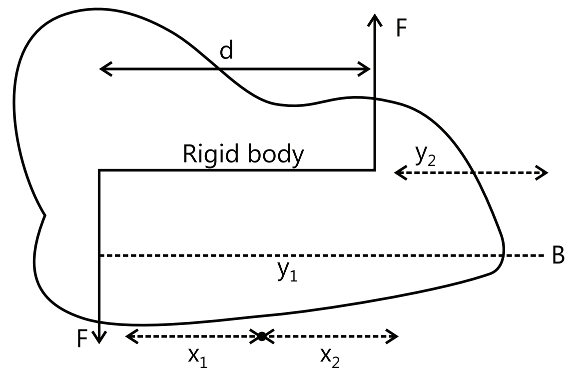


Figure 7.26: Force couple

6.4 Torque on a Rigid Body Executing Pure Rotation

Suppose I is the moment of inertia of a rigid body about the axis of rotation which is stationary in a given reference frame. The body is executing pure rotational motion about this fixed axis.

τ_{ext} = resultant torque about the axis of rotation due to all the external forces acting on the body

α = instantaneous angular acceleration of the body.

ω = instantaneous angular velocity of the body.

Consider one particle of the body say i^{th} particle of mass m_i at perpendicular distance r_i from the axis.

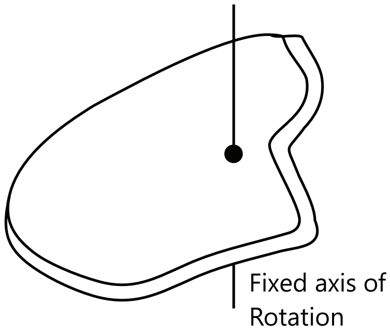


Figure 7.27: Rigid body executing pure rotation

Radial force on the particle $F_r = m\omega^2 r$ towards the center of its circular path.

Tangential force on the particle $F_t = m_i a_t = m_i \alpha r_i$

Torque of the radial force about the axis of rotation is zero as it intersects the axis. Torque of tangential force about the axis will be,

$$\tau_i = r_i F_t = m_i r_i^2 \alpha$$

To find the total torque on the rigid body about the axis we take summation of torques acting on all the particles of the body. The total torque comes out to be equal to the resultant torque due to external forces only as the torques due to internal forces cancel each other in pairs when summation is taken on all the particles of the body (By Newton's third law of motion internal forces form pairs of equal and opposite collinear forces. So the lever arms of the forces of a pair with respect to the axis will be equal so their torques will have equal magnitude but opposite directions and cancel each other in the summation). So

$$\tau_{ext} = \sum_i \tau_i = \left(\sum_i m_i r_i^2\right) \alpha = I\alpha \qquad \text{...(i)}$$

Remember: This formula is applicable only for pure rotational motion of a rigid body about a fixed axis.

7. KINETIC ENERGY OF BODY IN PURE ROTATION

When a rigid body performs pure rotational motion about a given axis, all of its constituent particles move in circular paths with centers on the axis and radii r_1, r_2 ...... and r_n (say), and with linear velocities $v_1 = \omega r_1$, $v_2 = \omega r_2$,........... and $v_n = \omega r_n$. If m_1, m_2,..... and m_n are the masses of the particles then the total kinetic energy of the rigid body is given by

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \text{.........} + \frac{1}{2} m_n v_n^2 \qquad \text{...(i)}$$

$$= \frac{1}{2} m_1\omega^2 r_1^2 + \frac{1}{2} m_2\omega^2 r_2^2 + ...+\frac{1}{2} m_n\omega^2 r_n^2$$

$$= \frac{1}{2}(m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2)\omega^2$$

Now as we have learnt the term $m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2$ is the moment of inertia of the rigid body.

Hence, the rotational kinetic energy of a body is given by

$$K = \frac{1}{2} I\omega^2 \quad ...(ii)$$

MASTERJEE CONCEPTS

Most of the problems involving incline and a rigid body, can be solved by using the conservation of energy. Care has to be taken in writing down the total Kinetic energy. Rotational Kinetic Energy term has to be taken into consideration along with translational kinetic energy. And while writing the rotational energy, the axis about which the moment of inertia is taken should be carefully chosen.

The point about which the conservation is done should be inertial to avoid calculating the work done by pseudo forces or the point itself should be the COM so that the work done by the torque of pseudo forces would be 0.

Shrikant Nagori (JEE 2009 AIR 30)

Illustration 16: A uniform circular disc has radius R and mass m. A particle, also of mass m, is fixed at a point A on the edge of the disc as shown in Fig 7.28. The disc can freely rotate about a fixed horizontal chord PQ that is at a distance R/4 from the center C of the disc. The line CA is ⊥ to PQ. Initially the disc is held vertical with point A at its highest point. It is then allowed to fall so that it starts rotating about PQ. Find the linear speed of the particle as it reaches lowest point. **(JEE ADVANCED)**

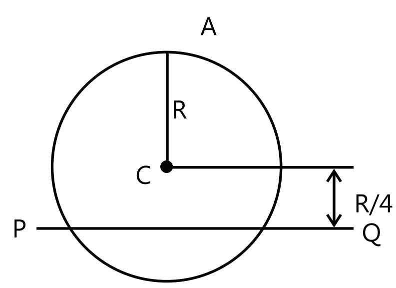


Figure 7.28

Sol: Find the moment of inertia of circular disc and the particle at point A about the chord PQ. The loss in potential energy of the system comprising the disc and the particle will be equal to the gain in its rotational kinetic energy.

$$I = \frac{1}{2} \times \frac{mR^2}{2} + m\left(\frac{R}{4}\right)^2 + m\left(\frac{5R}{4}\right)^2 = \frac{15mR^2}{8}$$

Energy equation

$$mg\frac{5R}{4} + \frac{mgR}{4} = \frac{1}{2}I\omega^2 - mg\frac{5R}{4} - \frac{mgR}{4}$$

$$\omega = 4\sqrt{\frac{g}{5R}}$$

$$V = \frac{5R\omega}{4} = \sqrt{5gR}$$

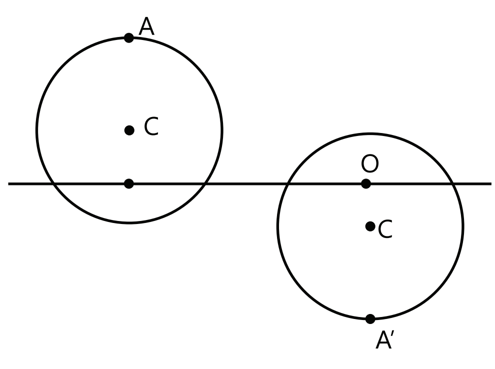


Figure 7.29

Illustrations 17: A pulley having radius r and moment of inertia I about its axis is fixed at the top of an inclined plane of inclination θ as shown in Fig 7.30. A string is wrapped round the pulley and its free end supports a block of mass m which can slide on the plane initially. The pulley is rotated at a speed ω_0 in a direction such that the block slides up the plane. Calculate the distance moved by the block before stopping? **(JEE ADVANCED)**

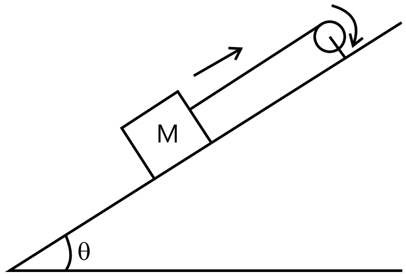


Figure 7.30

Sol: Apply Newton's second law of motion for block M along the inclined plane. Find the torque (about its axis) of force of tension acting on pulley. This will be equal to the product of moment of inertia I and the angular acceleration of pulley.

Suppose the deceleration of the block is a. The linear deceleration of the rim of the pulley is also a. The angular deceleration of the pulley is $\alpha = a/r$. If the tension in the string is T, the equations of motion are as follows:

$$mg\sin\theta - T = ma \quad \text{and} \quad Tr = I\alpha = Ia/r.$$

Eliminating T from these equations,

$$mg\sin\theta - I\frac{a}{r^2} = ma; \text{ Giving, } a = \frac{mgr^2\sin\theta}{I + mr^2}$$

The initial velocity of the block up the incline is $v = \omega_0 r$ Thus, the distance moved by the block before stopping is

$$x = \frac{v^2}{2a} = \frac{(I + mr^2)\omega_0^2}{2mg\sin\theta}$$

Illustration 18: A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H.

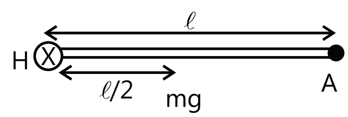


Figure 7.31

(i) Find angular acceleration α of the rod just after it is released from initial horizontal position from rest?

(ii) Calculate the acceleration (tangential and radial) of point A at this moment.

(iii) Calculate net hinge force acting at this moment.

(iv) Find α and ω when rod becomes vertical.

(v) Find hinge force when rod become vertical. **(JEE ADVANCED)**

Sol: The axis of rotation passing through H is fixed. So the torque of force of gravity about axis through H is equal to the product of moment of inertia about axis through H and angular acceleration of rod. Angular acceleration at an instant can be found if the torque of force of gravity at the instant is known.

(i) $\tau_H = I_H\ \alpha$

$$mg.\frac{\ell}{2} = \frac{m\ell^2}{3}\alpha \Rightarrow \alpha = \frac{3g}{2\ell}$$

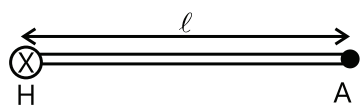


Figure 7.32

(ii) $a_{tA} = \alpha\ell = \frac{3g}{2\ell}.\ell = \frac{3g}{2}$

$a_{CA} = \omega^2 r = 0.\ell = 0$ ($\because \omega = 0$ just after release)

(iii) Suppose hinge exerts normal reaction in component form as shown

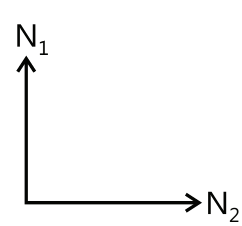


Figure 7.33

In vertical direction

$F_{ext} = ma_{CM}$

$$\Rightarrow mg - N_1 = m.\frac{3g}{4}$$

(We get the value of a_{CM} from previous example)

$$\Rightarrow N_1 = \frac{mg}{4}$$

In horizontal direction

$F_{ext} = ma_{CM} \Rightarrow N_2 = 0$ ($\because a_{CM}$ in horizontal = 0 as $\omega = 0$ just after release)

(iv) Torque = 0 when rod becomes vertical so $\alpha = 0$

Using energy conservation $\frac{mg\ell}{2} = \frac{1}{2}I\omega^2 \quad \left(I = \frac{m\ell^2}{3}\right)$

(Work done by gravity when COM moves down by (½) ℓ = change in K.E.)

$$\omega = \sqrt{\frac{3g}{\ell}}$$

(v) When rod becomes vertical

$\alpha = 0, \omega = \sqrt{\frac{3g}{\ell}}$ (Using $F_{net} = Ma_{CM}$)

$F_H - mg = \frac{m\omega^2\ell}{2}$ (a_{CM} = centripetal acceleration of COM)

Ans. $F_H = \frac{5mg}{2}$

Illustration 19: A bar of mass m is held as shown between 4 disks each of mass m′ and radius r = 75 mm. Determine the acceleration of the bar immediately after it has been released from rest, knowing that the normal forces exerted on the disks are sufficient to prevent any slipping and assuming that. (a) m = 5 kg and m′ = 2 kg.

(b) The mass m′ of the disks is negligible.

(c) The mass m of the bar is negligible **(JEE ADVANCED)**

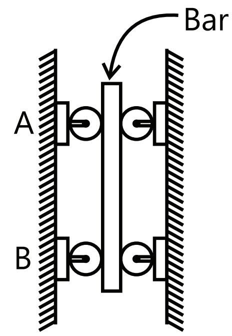


Figure 7.34

Sol: Apply Newton's second law of motion in vertical direction for the motion of center of mass of bar. Write the equation of torque due to force of friction acting on disc, for rotational motion about fixed axis through center of disk. Acceleration of rod will be equal to the tangential acceleration of the disc in the case of no slipping.

(a) Equation of center of mass of rod,

$mg - 4f = ma$ ....(i)

(where f is frictional force from one disk)

Torque acting on each disk due to frictional force is

$fr = \frac{m'r^2}{2}\frac{a}{r}$ ....(ii)

From (i) and (ii) we get

$mg - 2m'a = ma$ ....(iii)

$5g = (5 + 2 \times 2)a; \quad a = \frac{5g}{9}$

(b) Putting m′=0 in eqn. (iii) we get a = g

(c) Putting m = 0 in eqn. (iii) we get a = 0

(a) $\frac{5g}{9}\downarrow$

(b) $g\downarrow$ c) 0

7.1 Work Done and Power Delivered by Torque

If a torque τ rotates a body through an angle $d\theta$, the work, dW done by it is given by

$$dW = \tau . d\theta$$

The total work done W in rotating a body from the initial angle θ_1 to the final angle θ_2, is

$$W = \int_{\theta_1}^{\theta_2} \tau . d\theta = \int_{\omega_1}^{\omega_2} I \frac{d\omega}{dt} \omega dt = \int_{\omega_1}^{\omega_2} I\, \omega\, d\omega = \frac{1}{2} I\omega_2^2 - \frac{1}{2} I\omega_1^2$$

So the work done by torque is equal to the change in the rotational kinetic energy.

$$W = \Delta K_{rot} = \Delta\left(\frac{1}{2} I\omega^2\right) \quad ...(i)$$

This is called the **Work-Energy Theorem for rotation of rigid body**.

The rate at which work is done is called power P, given by

$$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega \quad ...(ii)$$

Also, the power P delivered by the torque on the rigid body is equal to the rate of change of kinetic energy

$$K = \frac{1}{2} I\omega^2 \qquad \therefore P = \frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2} I\omega^2\right)$$

$$\therefore P = \frac{1}{2} \times I \times 2\omega \frac{d\omega}{dt} = I \frac{d\omega}{dt}\, \omega = \tau\, \omega$$

8. EQUILIBRIUM OF RIGID BODIES

A rigid body can be in linear equilibrium as well as in rotational equilibrium. If a rigid body is in linear equilibrium, then the vector sum of all the forces acting on it should be zero.

i.e. $\Sigma\, \vec{F}_{ext} = 0$

Taking scalar components along the three axes x, y and z we get $\Sigma F_x = 0, \Sigma F_y = 0, \Sigma F_z = 0$

If a rigid body is in rotational equilibrium then the vector sum of all the external torques acting on it with respect to an axis in a given reference frame must be zero.

$$\Sigma\tau_{ext} = 0 \Rightarrow \Sigma\tau_x = 0, \Sigma\tau_y = 0, \Sigma\tau_z = 0$$

Illustrations 20: Two boys weighing 20 kg and 25 kg are trying to balance a seesaw of total length 4 m, with the fulcrum at the center. If one of the boys is sitting at an end, where should the other sit? **(JEE MAIN)**

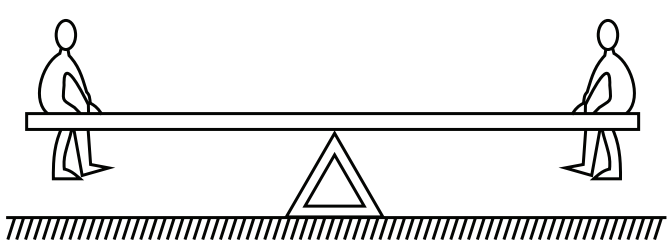


Figure 7.35

Sol: For rotational equilibrium, the net torque about the fulcrum of all the forces acting on the boys and the seesaw should be zero.

It is clear that the 20 kg kid should sit at the end and the 25 kg kid should sit closer to the center. Suppose his distance from the center is x. As the boys are in equilibrium, the normal force between a boy and the seesaw equals the weight of that boy. Considering the rotational equilibrium of the seesaw, the torque of the forces acting on it should add to zero. The forces are

(a) (25kg) g downward by the 25 kg boy

(b) (20kg) g downward by the 20 kg boy

(c) Weight of the seesaw and

(d) The normal force by the fulcrum.

Taking torques about the fulcrum.

(25 kg) g x = (20 kg) g (2 m) or x = 1.6 m

9. ANGULAR MOMENTUM

9.1 Angular Momentum of a Particle About a Point

If $\vec{p}$ is the linear momentum of a particle in a given reference frame, then angular momentum of the particle about an origin O in this reference frame is defined as

$$\vec{L} = \vec{r} \times \vec{p} \qquad \text{...(i)}$$

where $\vec{r}$ is the position vector of the particle with respect to origin O (see Fig. 7.36).

Magnitude of angular momentum is $L = rp\sin\theta$

or $L = r_{\perp}\, p$ or $L = p_{\perp}\, r$

θ = angle between vectors $\vec{r}$ and $\vec{p}$

$r_{\perp}$ = component of position vector $\vec{r}$ perpendicular to vector $\vec{p}$.

$p_{\perp}$ = component of vector $\vec{p}$ perpendicular to position vector $\vec{r}$.

SI unit angular momentum is kg m^2 s^{-1}.

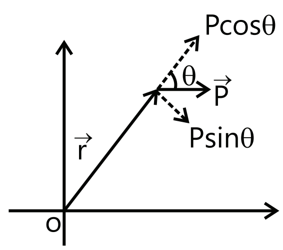


Figure 7.36: Angular momentum about a point

Relation between Torque and Angular Momentum

$\because \quad \vec{L} = \vec{r} \times \vec{p}$

Differentiating with respect to time we get

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times (m\vec{v}) + \vec{r} \times \vec{F} = 0 + \vec{r} \times \vec{F} = \vec{\tau}$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{\tau} \qquad \text{...(ii)}$$

For a single particle moving in a circle of radius r with angular velocity ω we have

$v = \omega r$ and $p = m\omega r$

So angular momentum comes out to be $L = r\,p = mr^2\omega$

Illustration 21: A particle of mass m is projected at time t = 0 from a point O with a speed u at an angle of 45° to the horizontal. Calculate the magnitude and direction of the angular momentum of the particle about the point O at time t = u/g. **(JEE ADVANCED)**

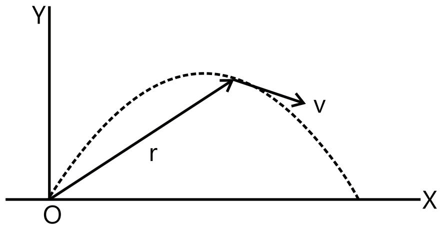


Figure 7.37

Sol: Express the position and velocity of particle in Cartesian coordinates in terms of unit vectors $\hat{i}$ and $\hat{j}$ and then calculate the cross product in

Cartesian coordinates.

Let us take the origin at O, X –horizontal axis and

Y – Axis along the vertical upward direction as shown in Fig 7.37 for horizontal during the time 0 to t.

$$v_x = u\cos 45° = u/\sqrt{2} \text{ and } x = v_x t = \frac{u}{\sqrt{2}}.\frac{u}{g} = \frac{u^2}{\sqrt{2}g}$$

For vertical motion,

$$v_y = u\sin 45° - gt = \frac{u}{\sqrt{2}} - u = \frac{(1-\sqrt{2})}{\sqrt{2}}u$$

and $$y = (u\sin 45°)\ t - \frac{1}{2}gt^2 = \frac{u^2}{\sqrt{2}g} - \frac{u^2}{2g} = \frac{u^2}{2g}(\sqrt{2}-1)$$

The angular momentum of the particle at time t about the origin is

$$\vec{L} = \vec{r} \times \vec{P} = m\vec{r} \times \vec{v} = m(\hat{i}x + \hat{j}y) \times (\hat{i}v_x + \hat{j}v_y) = m(\hat{k}xv_y - \hat{k}yv_x)$$

$$= m\,\hat{k}\left[\left(\frac{u^2}{\sqrt{2}g}\right)\frac{u}{\sqrt{2}}(1-\sqrt{2}) - \frac{u^2}{2g}(\sqrt{2}-1)\frac{u^2}{\sqrt{2}}\right] = -\hat{k}\frac{mu^3}{2\sqrt{2}g}$$

Thus, the angular momentum of the particle is $\frac{mu^3}{2\sqrt{2}g}$ in the negative z – direction i.e., perpendicular to the plane of motion, going into the plane.

Illustration 22: A cylinder is given angular velocity ω_0 and kept on a horizontal rough surface the initial velocity is zero. Find out distance travelled by the cylinder before it performs pure rolling and work by frictional force.

(JEE ADVANCED)

Sol: Due to backward slipping force of friction will act forwards. The cylinder is accelerated forwards. The torque due to friction and hence the angular acceleration is opposite to the initial angular velocity. So the angular velocity will decrease and the linear velocity of center of mass of cylinder will increase in the forward direction, till the slipping stops and pure rolling starts. The work done by frictional force is equal to change in the kinetic energy of the cylinder. The kinetic energy includes both rotational kinetic energy and translational kinetic energy.

$$\mu Mg\,R = \frac{MR^2\alpha}{2}$$

$$\alpha = \frac{2\mu g}{R} \quad \text{... (i)}$$

Initial velocity u = 0

$$v^2 = u^2 + 2as$$

$$v^2 = 2as \quad \text{... (ii)}$$

$$f_k = ma;\quad \mu Mg = Ma;\quad a = \mu g \quad \text{... (iii)}$$

$$\omega = \omega_0 - \alpha t$$

From equation (i) $\omega = \omega_0 - \frac{2\mu g}{R}t$; V = u + at

From equation (iii) $v = \mu\,g\,t$

$$\omega = \omega_0 - \frac{2v}{R};\ \omega = \omega_0 - 2\omega;\ \omega = \frac{\omega_0}{3}$$

From equation (ii)

$$\left(\frac{\omega_0 R}{3}\right)^2 = (2as) = 2\mu\,gs;\qquad S = \left(\frac{\omega_0^2R^2}{18\mu g}\right)$$

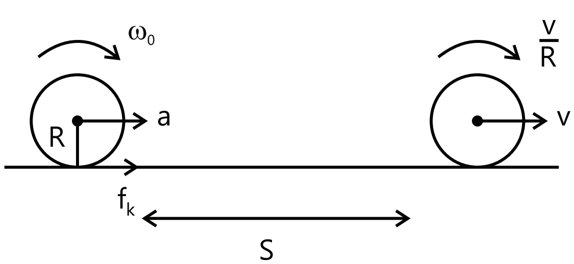


Figure 7.38

Work done by the frictional force

$$W = (-f_k R d\theta + f_k \Delta s) = -\mu mgR\Delta\theta + \frac{\mu mg \times \omega_0^2 R^2}{18\mu g};$$

$$\Delta\theta = \omega_0 \times t - \frac{1}{2}\alpha t^2 = \left\{\omega_0 \times \left(\frac{\omega_0 R}{3\mu g}\right)\frac{1}{2} \times \frac{2\mu g}{R}\left(\frac{\omega_0 R}{3\mu g}\right)^2\right\} = \left(\frac{\omega_0^2 R}{3\mu g} - \frac{\omega_0^2 R}{9\mu g}\right) = \frac{2\omega_0^2 R}{9\mu g}$$

$$W = \left\{\left(-\mu mg \times R\,\frac{2\omega_0^2 R}{9\mu g}\right) + \left(\mu mg \times \frac{\omega_0^2 R^2}{18\mu g}\right)\right\} = -\frac{m\omega_0^2 R^2}{6}$$

Illustration 23: A hollow sphere is projected horizontally along a rough surface with speed v and angular velocity ω_0. Find out the ratio $\frac{v}{\omega_0}$, so that the sphere stops moving after some time. **(JEE ADVANCED)**

Sol: For the sphere to stop after sometime, the acceleration should be opposite to velocity, i.e. the force of friction should be backwards (forward slipping). Also, the torque due to friction should be opposite to angular velocity, i.e. if the torque due to friction is clockwise (see Fig. 7.39), then the initial angular velocity should be anti-clockwise.

Torque about lowest point of sphere

$f_K \times R = I\alpha$; $\mu mg \times R = \frac{2}{3}mR^2\alpha$; $\alpha = \frac{3\omega g}{2R}$ (Angular acceleration in opposite direction of angular velocity)

$\omega = \omega_0 - \alpha t$ (Final angular velocity $\omega = 0$)

$\omega_0 = \frac{3\omega g}{2R} \times t$; $t = \frac{\omega_0 \times 2R}{3\text{ì}\, g}$

Acceleration $a = \mu g$

$v_t = v - at$ (Final velocity $v_t = 0$);

$v = \mu g \times t$; $t = \frac{v}{\mu g}$

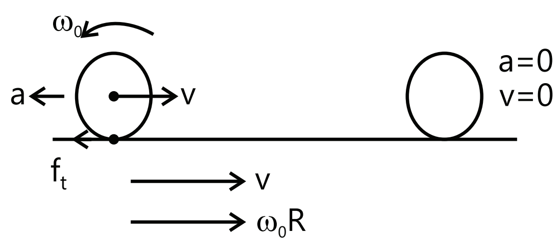


Figure 7.39

To stop the sphere, time at which v and ω are zero, should be same.

$$\frac{v}{\mu g} = \frac{2\omega_0 R}{3\text{ì}\, g}; \Rightarrow \frac{v}{\omega_0} = \frac{2R}{3}$$

Illustration 24: A rod AB of mass 2m and length ℓ is lying on a horizontal frictionless surface. A particle of mass m travelling along the surface hits the end of the rod with a velocity v_0 in a direction perpendicular to AB. The collision is elastic. After the collision the particle comes to rest. Find out after collision

(a) Velocity of center of mass of rod

(b) Angular velocity. **(JEE ADVANCED)**

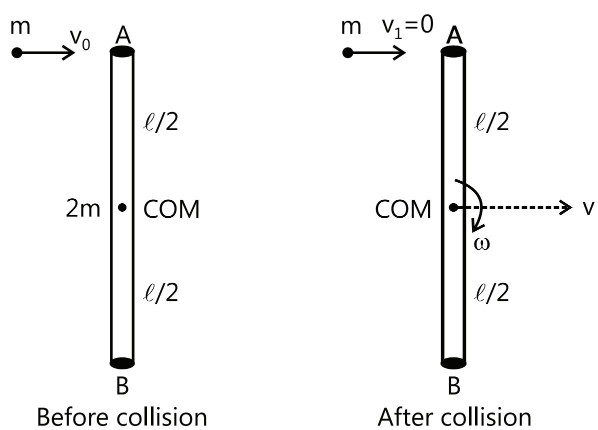


Figure 7.40

Sol: Conserve linear momentum and angular momentum of the system constituting "the rod and the particle" before and after collision. Here the linear and angular momentum of the rod before collision is zero. Angular momenta of the rod and particle are calculated about the center of the rod.

(a) Let just after collision the speed of COM of rod is v and angular velocity about COM is ω.

External force on the system (rod + mass) in horizontal plane is zero.

Apply conservation of linear momentum in x direction;

$$mv_0 = 2mv \qquad \text{..... (i)}$$

(b) Net torque on the system about any point is zero

Apply conservation of angular momentum about COM of rod.

$$mv_0\frac{\ell}{2} = I\omega \Rightarrow mv_0\frac{\ell}{2} = \frac{2m\ell^2}{12}\omega \qquad \text{...... (ii)}$$

From equation (i) velocity of center of mass $v = \frac{v_0}{2}$

From equation (ii) angular velocity $\omega = \frac{3v_0}{\ell}$.

9.2 Angular Momentum of a Rigid Body Rotating About Fixed Axis

For a system of particles the total angular momentum about an origin is the sum of the angular momenta of all the particles calculated about the same origin.

$$\vec{L} = \sum_i \vec{L}_i$$

Differentiating with respect to time we get,

$$\frac{d\vec{L}}{dt} = \sum_i \frac{d\vec{L}_i}{dt} = \sum_i\left(\sum_k \vec{\tau}_{ik} + \vec{\tau}_i^{ext}\right) = \sum_i\sum_k \vec{\tau}_{ik} + \sum_i \vec{\tau}_i^{ext} = 0 + \vec{\tau}^{ext}$$

The double summation term corresponds to the sum of torques due to internal forces and as explained earlier, according to Newton's third law of motion these internal torques cancel out in pairs.

So for a system of particles

$$\frac{d\vec{L}}{dt} = \vec{\tau}^{ext} \qquad \text{...(xvii)}$$

Impulse of a torque is defined as $J = \int d\vec{L} = \int \vec{\tau}^{ext} dt$

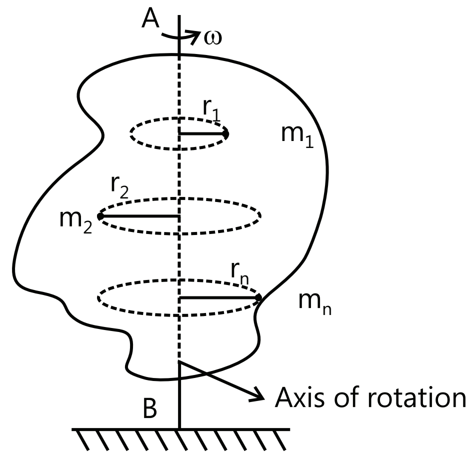


Figure 7.41: Angular momentum of rigid body

Angular momentum of a rigid body rotating about a fixed axis can be calculated as below:

Angular momenta of its individual particles about the axis are

$L_1 = m_1r_1^2\omega$, $L_2 = m_2r_2^2\omega$, $L_3 = m_3r_3^2\omega$, ... $L_n = m_nr_n^2\omega$ where ω is the instantaneous angular velocity of the rigid body

Total angular momentum of the body

$$L = m_1r_1^2\omega + m_2r_2^2\omega + m_3r_3^2\omega \text{ ....} + m_nr_n^2\omega$$

$$L = \sum_i m_i(r_i)^2\omega = I\,\omega$$

So $L = I\,\omega$

Remember: This formula is applicable only for rotation of the rigid body about a fixed axis.

Again differentiating this relation with respect to time we get,

$$\frac{dL}{dt} = I\frac{d\omega}{dt} = I\alpha = \tau_{ext}$$

Illustration 25: Two small balls of mass m each are attached to a light rod of length ℓ, one at its center and the other at its free end. The rod is fixed at the other end and is rotated in horizontal plane at an angular speed ω. Calculate the angular momentum of the ball at the end with respect to the ball at the center. **(JEE MAIN)**

Sol: Both the balls A and B have same angular velocity but different linear velocities.

The situation is shown in Fig 7.42. The velocity of the ball A with respect to the fixed end O is $v_A = \omega(\ell/2)$ and that of B with respect to O is $v_B = \omega\ell$. Hence the velocity of B with respect to A is $v_B - v_A = \omega(\ell/2)$. The angular momentum of B with respect to A is, therefore,

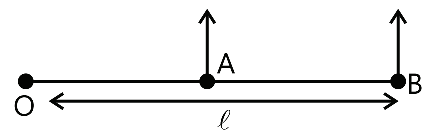


Figure 7.42

$$L = mvr = m\omega\left(\frac{\ell}{2}\right)\frac{\ell}{2} = \frac{1}{4}m\omega\ell^2$$

along the direction perpendicular to the plane of rotation.

9.3 Conservation of Angular Momentum

In the previous article we have proved the relation

$$\frac{d\vec{L}}{dt} = \vec{\tau}^{ext} \text{ where } \vec{L} \text{ and } \vec{\tau}^{ext} \text{ are evaluated about the same origin.}$$

From the above equation we see that if $\vec{\tau}^{ext} = 0$ then $\vec{L}$ of the system of particles remains constant.

In some situations the component of external torque about an axis is zero even if the net external torque is not zero. So in these cases the component of the total angular momentum, about the particular axis, remains constant.

Illustration 26: A uniform rod of mass m and length ℓ can rotate freely on a smooth horizontal plane about a vertical axis hinged at point H. A point mass having same mass m coming with an initial speed u perpendicular to the rod strikes the rod in-elastically at its free end. Find out the angular velocity of the rod just after collision?

(JEE MAIN)

Sol: After collision the rod and the particle execute pure rotational motion about vertical axis through fixed point H.

Angular momentum is conserved about H because no external force is present in horizontal plane which is producing torque about H.

$$mu\ell = \left(\frac{m\ell^2}{3} + m\ell^2\right)\omega \quad \Rightarrow \quad \omega = \frac{3u}{4\ell}$$

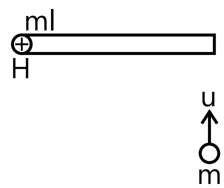


Figure 7.43

Illustration 27: A uniform rod of mass m_1 and length ℓ lies on a frictionless horizontal plane. A particle of mass m_2 moving at a speed v_0 perpendicular to the length of the rod strikes it at a distance $\ell/3$ from the center and stops after the collision. Calculate (a) the velocity of the center of the rod and (b) the angular velocity of the rod about its center just after the collision. **(JEE ADVANCED)**

Sol: Conserve the linear momentum of the system comprising "the rod and the particle" before and after the collision. Conserve the angular momentum, about the center of the rod, of the system comprising "the rod and the particle" before and after the collision.

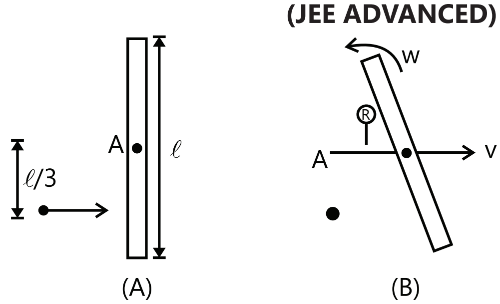


Figure 7.44

The situation is shown in the Fig 7.44. Consider the rod and the particle together as the system. As there is no external resultant force, the linear momentum of the system will remain constant. Also there is no resultant external torque on the system and so the resultant external torque on the system and the angular momentum of the system about the line will remains constant. Suppose the velocity of the center of the rod is V and the angular velocity about the center is ω .

(a) The linear momentum before the collision is m_2v_0 and that after the collision is M_1V.

Thus $m_2v_0 = m_1V$, or $V = \left(\frac{m_2}{m_1}\right)v_0$

(b) Let A be the center of the rod when at rest. Let AB be the line perpendicular to the plane of the Fig 7.44. Consider the angular momentum of N "the rod plus the particle" system about AB.

Initially the rod is at rest. The angular momentum of the particle about AB is

$$L = m_2v_0(\ell/3)$$

After collision the particle comes to rest. The angular momentum of the rod about a is

$$\vec{L} = \vec{L}_{CM} + m_1\vec{r_0} \times \vec{V}$$

As $\vec{r_0} \parallel \vec{V}$, $\vec{r_0} \times \vec{V} = 0$ thus, $\vec{L} = \vec{L}_{CM}$

Hence the angular momentum of the rod about AB is

$L = I\omega = \frac{m_1\ell^2}{12}\omega$ Thus, $\frac{m_2v\ell}{3} = \frac{m_1\ell^2}{12}\omega$ Or $\omega = \frac{4m_2v_0}{m_1\ell}$

10. RIGID BODY IN COMBINED TRANSLATIONAL AND ROTATIONAL MOTION

As discussed earlier, in this type of motion the rigid body is performing pure rotational motion about an axis and the axis itself is performing pure translational motion relative to a given reference frame.

Consider a car moving over a straight horizontal road with some instantaneous velocity v with respect to a reference frame K fixed to the road. Now let us observe the motion of a wheel of the car from the K frame. This motion of the wheel in K frame is an example of combined translational and rotational motion. Let us suppose a reference frame K' which is translating with respect to frame K with same instantaneous velocity v. In other words frame K' is rigidly fixed to the body of the car. In this frame the wheel of the car performs pure rotational motion. The body of the car itself is performing pure translational motion.

Take another example of motion of a fan fixed inside the car.

If the fan is switched off while the car is moving on the road, the motion of fan is **pure translational** with respect to K frame.

If the fan is switched on while the car is at rest, the motion of fan is **pure rotational** about its axis, as the axis is at rest in the K frame.

If the fan is switched on while the car is moving on the road, the motion of the fan with respect to K frame is neither pure translational nor pure rotational but a combination of both. Now if an observer A is sitting inside the car, as the car moves, the motion of fan will appear to him as pure rotational while the motion of the observer A with respect to K frame is pure translational. Hence in this case we can see that the motion of the fan can be resolved into two components, pure rotational motion relative to observer A and pure translational motion of observer A relative to K frame.

Such a resolution of motion of a rigid body into components of pure rotational and pure translational motion is an important tool used in the study of their dynamics.

10.1 Kinematics of a General Rigid Body Motion

For a rigid body the value of angular displacement θ, angular velocity ω, and angular acceleration α is same for all points on the rigid body. Also, if we choose any point of the rigid body as origin O and any other point, say A, of the body has a position vector $\vec{r}$ relative to O, and during any time interval the vector $\vec{r}$ rotates by an angle θ relative to its initial direction, then position vector of any other point, say B, relative to any other origin, say O′, inside the rigid body will also rotate by the same angle θ. This means the angular variables θ, ω, and α do not depend on the choice of origin in the rigid body.

The above concept is very important as it enables us to calculate the velocity of each point of the rigid body if we know the velocity of any one point (say A) in the rigid body with respect to a reference frame K and angular velocity of any point in the rigid body relative to any other point in the rigid body.

Suppose we want to calculate the velocity of a point B in the rigid body which has a position vector $\vec{r}_{BA}$ relative to A (see Fig. 7.45).

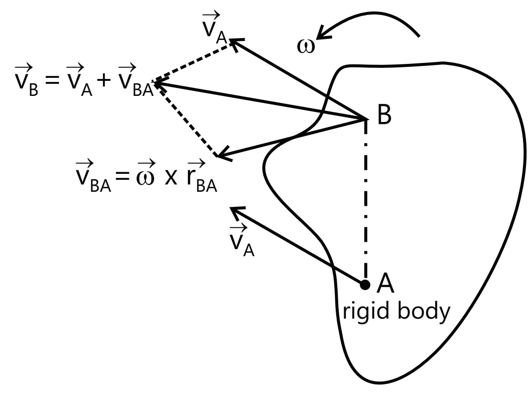


Figure 7.45: Kinematics of rigid body

The velocity of point A is $\vec{v}_A$, so we have velocity of B as

$$\vec{v}_B = \vec{v}_A + \vec{v}_{BA} = \vec{v}_A + \vec{\omega} \times \vec{r}_{BA}$$

Direction of $\vec{\omega}$ is given by right hand thumb rule. If we curl the fingers of the right hand in the direction of rotation of the body, thumb gives the direction of $\vec{\omega}$.

Similarly the acceleration of point B is: $\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{BA}$

Illustration 28: Consider the general motion of a wheel (radius r) which can be viewed as pure translation of its center O (with the velocity v) and pure rotation about O (with angular velocity ω)

Find out $\vec{v}_{AO}, \vec{v}_{BO}, \vec{v}_{CO}, \vec{v}_{DO}$ and $\vec{v}_A, \vec{v}_B, \vec{v}_C, \vec{v}_D$ **(JEE MAIN)**

Sol: Express the angular velocity, linear velocity of point O and position vectors of points A, B and C relative to O in Cartesian coordinates.

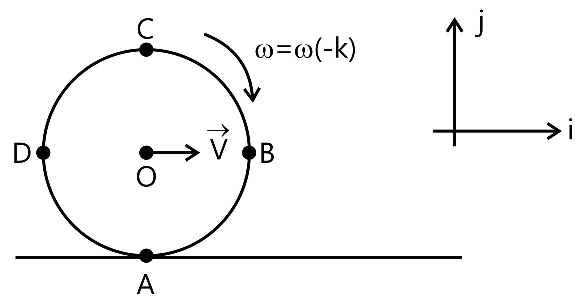


Figure 7.46

$$\vec{v}_{AO} = (\vec{\omega} \times \vec{r}_{AO}) = \left(\omega(-\hat{k}) \times \vec{OA}\right)$$

$$= \left(\omega(-\hat{k}) \times r(-\hat{j})\right) = -\omega r\hat{i}$$

Similarly $\vec{v}_{BO} = \omega r\,(-\hat{j})$; $\vec{v}_{CO} = \omega r(\hat{i})$; $\vec{v}_{DO} = \omega r(\hat{j})$

$$\vec{v}_A = \vec{v}_O + \vec{v}_{AO} = v\hat{i} - \omega r\hat{i} \;;$$

$$\vec{v}_B = \vec{v}_O + \vec{v}_{BO} = v\hat{i} + \omega r\hat{j}$$

$$\vec{v}_C = \vec{v}_O + \vec{v}_{CO} = v\hat{i} + \omega r\hat{i} \;; \qquad \vec{v}_D = \vec{v}_O + \vec{v}_{DO} = v\hat{i} + \omega r\hat{j}$$

10.2 Dynamics of a General Rigid Body Motion

Combined rotation and translation of a rigid body is considered as combination of pure rotation in C frame about an axis passing through the center of mass and translation of center of mass in a reference frame K. Dynamics of combined rotational and translational motion of a rigid body in K frame is defined by two vector equations. One of them describes the dynamics of the center of mass of the rigid body in the K frame, and the other the equation of dynamics of pure rotation of the body about center of mass in the C frame.

So if the total mass of the rigid body is M and the net external force acting on it is $\vec{F}_{ext}$ then we have in the K frame,

$$M\frac{d\vec{V}_C}{dt} = \vec{F}_{ext} \qquad \text{...(i)}$$

If I_C is the moment of inertia of the rigid body about the axis passing through center of mass and $\vec{\tau}_C$ is the net torque of all external forces about the axis passing through the center of mass, then we have in the C frame,

$$\vec{\tau}_C = I_C\frac{d\vec{\omega}}{dt} = I_C\vec{\alpha} \qquad \text{...(ii)}$$

If $\vec{P}_{total}$ is the total linear momentum of the rigid body in the K frame, $\vec{L}_C$ is angular momentum of the body in C frame about center of mass and $\vec{r}_C$ is the position vector of center of mass relative to some origin in K frame, then we have,

$$\vec{P}_{total} = M\vec{V}_C$$

Total Kinetic energy

$$K = \frac{1}{2}MV_C^2 + \frac{1}{2}I_C\omega^2 \quad \text{...(iii)}$$

$$\vec{L}_C = I_C\ \vec{\omega} \quad \text{...(iv)}$$

Angular momentum in K frame = $\vec{L}_C$ about C.O.M + $\vec{L}$ of the C.O.M about some origin in K frame

$$\vec{L} = \vec{I}_C\ \vec{\omega} + \vec{r}_C \times M\vec{V}_C \quad \text{...(v)}$$

10.3 Pure Rolling (Rolling Without Slipping)

Pure rolling is a special case of combined translational and rotational motion of a rigid body with circular cross section (e.g. wheel, disc, ring, cylinder, sphere etc.) moving on a surface. Here, there is no slipping between the rolling body and the surface at the point of contact.

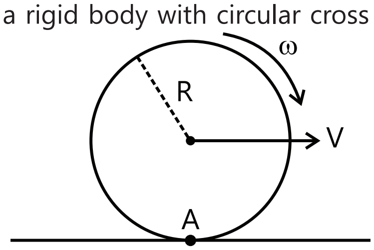


Figure 7.47: Sphere rolling on a stationary surface.

Suppose a sphere rolls on a stationary surface and the point of contact between the sphere and the surface is A (see Fig. 7.47). Let the velocity of the center of sphere be v, radius be R and its angular velocity be ω. For pure rolling the relative velocity between the point A of the sphere and the surface must be zero. As the surface is at rest, the velocity of point A is also zero.

$\therefore v_A = v - \omega R = 0$

$\therefore v = \omega R$

If sphere is rolling on a plank moving velocity v_0, then for pure rolling, $v_A = v - \omega R = v_0$ (see Fig. 7.48)

Same is true for the tangential acceleration of the point of contact in case of pure rolling.

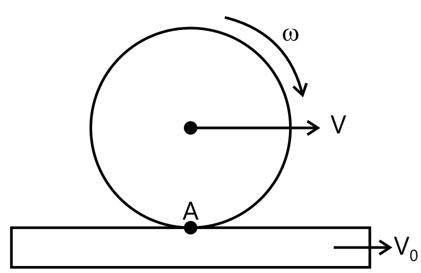


Figure 7.48: Sphere rolling on a moving surface.

Now let's discuss the case where a rolling cylinder of mass m moves forward on a rough plate of same mass with acceleration "a" and the rough plate moves forward with an acceleration "a_0" under action of force F on a smooth surface.

As the cylinder accelerates in the forward direction, so by Newton's second law, the friction on the cylinder at the point of contact will be in forward direction and on the plate in backward direction by Newton's third law (see Fig. 7.50).

Equation of torque about center of cylinder:

$$\tau_C = fR = \frac{mR^2}{2}\alpha$$

$$\Rightarrow \alpha = \frac{2f}{mR} \quad \text{....... (i)}$$

Equation of motion of center of cylinder:

$$f = ma \quad \text{...... (ii)}$$

From (i) and (ii) we get

$$a = \frac{\alpha R}{2}$$

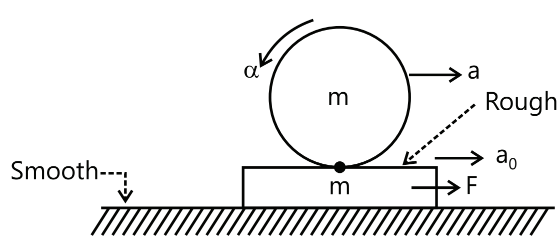


Figure 7.49: Cylinder rolling on an accelerating plate.

At contact point

$\alpha_0 = a + \alpha R = \frac{3\alpha R}{2} = 3a$ ...... (iii)

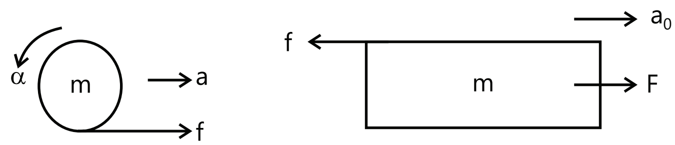


Figure 7.50: (a) FBD of Cylinder. (b) FBD of Plate.

Equation of motion of plate:

$F - f = ma_0$

$F = m(a + a_0)$

$F = 4ma$; $a = \frac{F}{4m}$; $a_0 = \frac{3F}{4m}$

Illustration 29: A wheel of radius r rolls (rolling without slipping) on a level road as shown in fig 7.51.

Find out velocity of point A and B. **(JEE MAIN)**

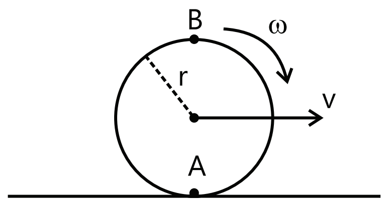


Figure 7.51

Sol: Linear velocity of any point on the rim of the wheel has magnitude ωr in the reference frame of center of wheel (C-frame). Velocity in ground frame is the vector sum of velocity in C-frame and the velocity of center of wheel.

Contact point at surface is in rest for pure rolling

Velocity of point is A zero.

So $v = \omega r$

Velocity of point B = $v + \omega r = 2v$

Illustration 30: A uniform sphere of mass 200 g rolls without slipping on a plane surface so that its center moves at a speed of 2.00 cm s^{-1}. Find its kinetic energy. **(JEE MAIN)**

Sol: The kinetic energy of sphere is the sum of the translational kinetic energy and the rotational kinetic energy.

As the sphere rolls without slipping on the plane surface its angular speed about center is

$\omega = \frac{v_{cm}}{r}$. The kinetic energy is $K = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2}Mv_{cm}^2 = \frac{1}{2}.\frac{2}{5}Mr^2\omega^2 + \frac{1}{2}Mv_{cm}^2$

$$= \frac{1}{5}Mv_{cm}^2 + \frac{1}{2}Mv_{cm}^2 = \frac{7}{10}Mv_{cm}^2 = \frac{7}{10}(0.200\text{ kg})(0.02\text{ m s}^{-1})^2 = 5.6 \times 10^{-5}\text{ J}$$

Illustration 31: A constant force F acts tangentially at the highest point of a uniform disc of mass m kept on a rough horizontal surface as shown in Fig 7.52. If the disc rolls without slipping, calculate the acceleration of the Center C and point A and B of the disc. **(JEE ADVANCED)**

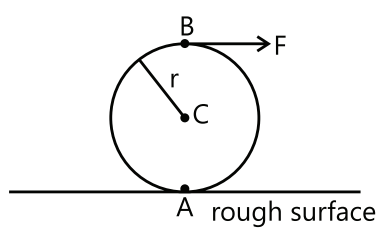


Figure 7.52

Sol: Apply Newton's second law for the motion of center of mass of the disc. Find the torque of the force F and the force of friction acting on the disc at point A about the center of mass of the disc and thus obtain the equation relating the angular acceleration in the C-frame to the torques of all the external forces.

The situation is shown in Fig 7.52. As the force F rotates the disc, the point of contact has a tendency to slip towards left so that the static friction on the disc will act towards right. Let r be the radius of the disc and be the linear acceleration of the center of the disc. The angular acceleration about the center of the disc is

$\alpha = a/r$, as there is no slipping.

For the linear motion of the center,

$F + f = ma$ .... (i)

And for the rotation motion about the center,

$$Fr - fr = I\alpha = Fr - fr = I = \left(\frac{1}{2}mr^2\right)\left(\frac{a}{r}\right) \text{ or } F - f = \frac{1}{2}ma$$ .... (ii)

From (i) and (ii),

$$2F = \frac{3}{2}ma \quad \text{or} \quad a = \frac{4F}{3m}$$

Acceleration of point A is zero

Acceleration of point B is $2a = 2\left(\frac{4F}{3m}\right) = \left(\frac{8F}{3m}\right)$.

Illustration 32: A circular rigid body of mass m, radius R and radius of gyration (k) rolls without slipping on an inclined plane of an inclination θ. Find the linear acceleration of the rigid body and force of friction on it. What must be the minimum value of coefficient of friction so that rigid body may roll without sliding?

(JEE ADVANCED)

Sol: Apply Newton's second law for the motion of center of mass of the rigid body. Find the torque of the force F and the force of friction acting on the rigid body about the center of mass of the disc and thus obtain the equation relating the angular acceleration in the C-frame to the torques of all the external forces.

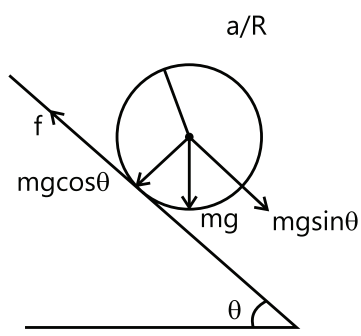


Figure 7.53

If a is the acceleration of the center of mass of the rigid body and f the force of friction between sphere and the plane, the equation of translational and rotational motion of the rigid body will be

$Mg\sin\theta - f = ma$ (Translational motion)

$fR = I\alpha$ (Rotational motion)

$$f = \frac{I\alpha}{R} \qquad I = mk^2, \text{ due to pure rolling } a = \alpha R$$

$$mg\sin\theta - \frac{I\alpha}{R} = m\alpha R = m\alpha R + \frac{I\alpha}{R} = ma + \frac{mk^2\alpha}{R} = a\left[\frac{R^2 + k^2}{R^2}\right]$$

$$a = \frac{g\sin\theta}{\left(\frac{R^2+k^2}{R^2}\right)} = \frac{g\sin\theta}{\left(1+\frac{k^2}{R^2}\right)}; \qquad f = \frac{I\alpha}{R} = \frac{mk^2a}{R^2} \Rightarrow \frac{mg\,k^2\sin\theta}{R^2+k^2}$$

$$f \le \mu N; \qquad \frac{mk^2}{R^2}a \le \mu \le mg\cos\theta$$

$$R^2\frac{k^2}{R^2} \times \frac{g\sin\theta}{\left(k^2+R^2\right)} \le \mu g\cos\theta;\ \mu \ge \frac{\tan\theta}{\left[1+\frac{R^2}{k^2}\right]}; \qquad \mu_{min} \ge \frac{\tan\theta}{\left[1+\frac{R^2}{k^2}\right]}$$

MASTERJEE CONCEPTS

- From above example if rigid bodies are solid cylinder, hollow cylinder, solid sphere and hollow sphere (having radius 'r' and mass 'm')
- Increasing order of acceleration

  $a_{solid\ sphere} > a_{hollow\ sphere} > a_{solid\ cylinder} > a_{hollow\ cylinder}$

- Increasing order of required friction force for pure rolling

  $f_{hollow\ cylinder} > f_{hollow\ sphere} > f_{solid\ cylinder} > f_{solid\ sphere}$

- Increasing order of required minimum friction coefficient for pure rolling

  $\mu_{hollow\ cylinder} > \mu_{hollow\ sphere} > \mu_{solid\ cylinder} > \mu_{solid\ sphere}$

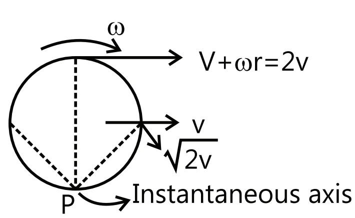
- I would advise you to derive these, verify and remember!

Anand K (JEE 2011 AIR 47)

10.4 Instantaneous Axis of Rotation

The combined translational and rotational motion of a rigid body can be reduced to a purely rotational motion. When we know the velocity V_C of the center of mass and the instantaneous angular velocity ω of the body then we can find a point whose velocity comes out to be zero at a given moment of time. The axis passing through this point at the given moment is called instantaneous axis of rotation and the rigid body performs pure rotation about this axis with same angular velocity at that moment.

The position of the instantaneous axis of rotation changes with time. E.g. in pure rolling the point of contact with the surface is the instantaneous axis of rotation (see Fig. 7.54).



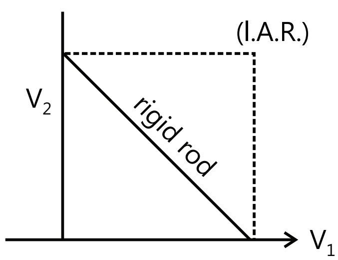


Figure 7.54: IAR (a) pure rolling; (b) Rod slipping down a wall

Geometrical construction of instantaneous axis of rotation (I.A.R). If we know the velocity vectors of any two points in the rigid body then the I.A.R. is the axis passing through the point of intersection of the perpendiculars drawn to the velocity vectors at those points.

Once location of I.A.R is known, we find the moment of inertia of the body about this axis, and then the equations of rotation about fixed axis can be used for this axis.

Illustration 33: Prove that kinetic energy = $1/2\ I_P\omega^2$ **(JEE MAIN)**

Sol: Kinetic energy is the sum of the translational kinetic energy and the rotational kinetic energy.

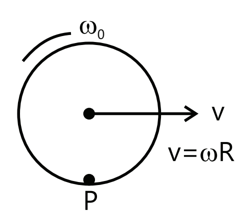


Figure 7.55

$$K.E. = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} Mv_{cm}^2 = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2} M\omega^2R^2 = \frac{1}{2}(I_{cm} + MR^2)\omega^2 = \frac{1}{2}\left(I_{contact\ point}\right)\omega^2$$

Notice that in pure rolling of uniform object, equation of torque can also be applied about the contact point.

Illustration 34: A uniform bar of length ℓ and mass m stands vertically touching a vertical wall (y – axis). When slightly displaced, its lower end begins to slide along the floor (x – axis). Obtain an expression for the angular velocity (ω) of the bar as a function of θ. Neglect friction everywhere.

(JEE ADVANCED)

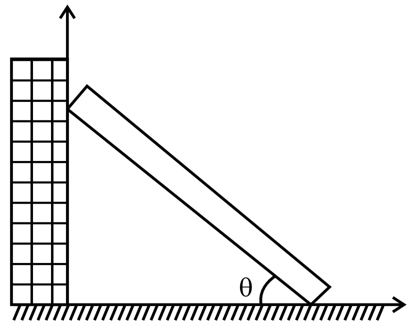


Figure 7.56

Sol: As the rod falls, it executes pure rotational motion about the instantaneous axis of rotation. The loss in gravitational potential energy is equal to the gain in the rotational kinetic energy.

The position of instantaneous axis of rotation (IAOR) is shown in Fig 7.57.

$C = \left(\frac{\ell}{2}\cos\theta, \frac{\ell}{2}\sin\theta\right)$; $\quad r = \frac{\ell}{2}$ = half of the diagonal

All surfaces are smooth, therefore, mechanical energy will remain conserved.

∴ Decrease in gravitational potential energy of bar = increase in rotational kinetic energy of bar about IAOR.

$$mg\frac{\ell}{2}(1 - \sin\theta) = \frac{1}{2}I\omega^2 \quad \text{... (i)}$$

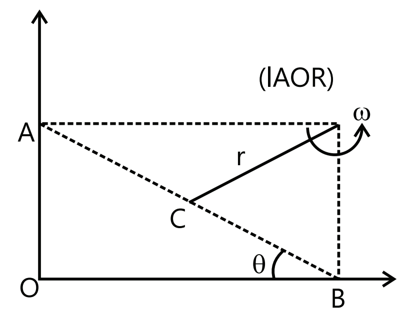


Figure 7.57

Here, $I = \frac{m\ell^2}{12} + mr^2$ (about IAOR) or I

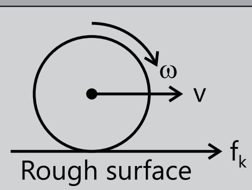
$= \frac{m\ell^2}{12} + \frac{m\ell^2}{4} = \frac{m\ell^2}{3}$ Substituting in Eq. (i)

We have $mg\frac{\ell}{2}(1 - \sin\theta) = \frac{1}{2}\left(\frac{m\ell^2}{3}\right)\omega^2$ or $\omega = \sqrt{\frac{3g(1-\sin\theta)}{\ell}}$

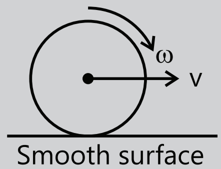
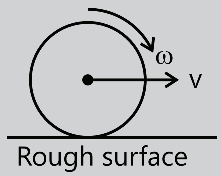
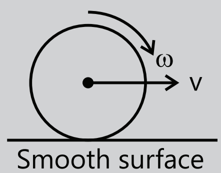
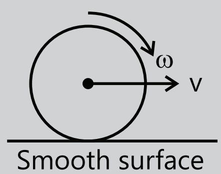
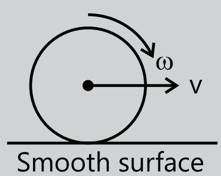
MASTERJEE CONCEPTS

Nature of friction for rigid bodies:

- A rigid body rolling with a speed of v and angular velocity of ω at an instant. Then it falls under one of the following cases.

Cases	Rough/Smooth	Diagram	Inference
$V < r\omega$	Rough Surface		1. There is relative motion at point of contact. With respect to the body the surface moves slower than itself. So the surface tries to decrease its angular velocity by a frictional force in forward direction. And this friction is kinetic friction. 2. It increases v and decreases ω So, after sometime, $v = r\omega$ and pure rolling will resume.

MASTERJEE CONCEPTS

Cases	Rough/Smooth	Diagram	Inference
	Smooth Surface		No friction is possible and it is not pure rolling.
$v > r\omega$	Rough surface		With respect to the COM of the cylinder, the surface moves at a higher speed than itself. So the surface tries to increase its angular velocity by exerting a frictional force in backward direction. And this friction would be kinetic friction. 2. The friction tries to reduce V and increase ω
$V > r\omega$	Smooth Surface		No friction and no pure rolling.
$V = r\omega$	Rough Surface		This is pure rolling. However there might be static friction acting on the body.
	Smooth Surface		No friction is possible and it is pure rolling

Rohit Kumar (JEE 2012 AIR 79)

Illustration 35: A rigid body of mass m and radius r rolls without slipping on a surface. A force is acting on the rigid body at x distance from the center as shown in Fig 7.58. Find the value of x so that static friction is zero.

(JEE MAIN)

Sol: For static friction to be zero, the linear and angular accelerations a and α caused by the force F should be related as a = αR, for rolling without slipping.

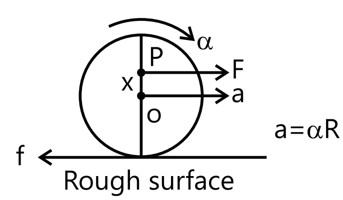


Figure 7.58

Torque about center of mass $Fx = I_{cm}\alpha$ ... (i)

$F = ma$ ... (ii)

From equation (i) & (ii) $\quad max = I_{cm}\alpha \quad (a = \alpha R)$;

$$x = \frac{I_{cm}}{mR}$$

Illustration 36: There are two cylinders of radii R_1 and R_2 having moments of inertia I_1 and I_2 about their respective axes as shown in Fig 7.59. Initially, the cylinders rotate about their axes with angular speed ω_1 and ω_2 as shown in the Fig 7.59. The cylinders are moved close to touch each other keeping the axes parallel. The cylinders first slip over each other at the contact but the slipping finally ceases due to the friction between them. Calculate the angular speeds of the cylinders after the slipping ceases. **(JEE ADVANCED)**

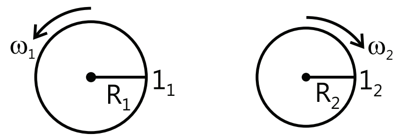


Figure 7.59

Sol: The force of friction acting on the cylinder moving faster will be such that its angular velocity decreases. The force of friction acting on the cylinder moving slower will be such that its angular velocity increases. When slipping ceases, the linear speeds of the points of contact of the two cylinders will be equal.

If ω'_1 and ω'_2 be the respective angular speeds at the instant slipping ceases, we have

$$\omega'_1 R_1 = \omega'_2 R_2 \quad \text{...(i)}$$

The change in the angular speed is brought about by the frictional force which acts as long as the slipping exists. If this force f acts for a time t. the torque on the first cylinder is fR_1 and that on the second is fR_2. Assuming $\omega_1 > \omega_2$. The corresponding angular impulses are $-fR_1 t$ and $fR_2 t$,

We therefore, have

$$-fR_1 t = I_1(\omega'_1 - \omega_1) \text{ and } fR_2 t = I_2(\omega'_2 - \omega_2)$$

$$\text{or } -\frac{I_1}{R_1}(\omega'_1 - \omega_1) = \frac{I_2}{R_2}(\omega'_2 - \omega_2) \quad \text{...(ii)}$$

Solving (i) and (ii) $\omega'_1 = \dfrac{I_1\omega_1 R_2 + I_2\omega_2 R_1}{I_2 R_1^2 + I_1 R_2^2} R_2$ and $\omega'_2 = \dfrac{I_1\omega_1 R_2 + I_2\omega_2 R_1}{I_2 R_1^2 + I_1 R_2^2} R_1$.

10.5 Energy Method in Solving Problems of Rolling Body

We can conserve energy in case of pure rolling of a rigid body because the point of contact between the surfaces remains at rest and so the frictional forces acting at the point of contact do not do any work. Thus only conservative force do work on the body.

Thus Potential energy + total K.E. = constant

As shown in the Fig. 7.60, a disc is rolling down on an inclined plane. Then we can conserve total mechanical energy. If the disc falls a height h then loss in potential energy is equal to gain in kinetic energy.

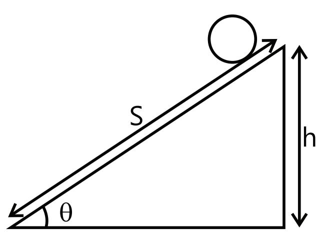


Figure 7.60

$$Mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}MV_C^2 \quad \text{... (i)}$$

$$\text{Its total kinetic energy} = \frac{1}{2}MV_C^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}MV_C^2\left(1+\frac{K^2}{R^2}\right) \quad \text{...(ii)}$$

where K is the radius of gyration of the disc and V_C the velocity of center of mass.

$$\text{So } \frac{1}{2}MV_C^2\left(1+\frac{K^2}{R^2}\right) = Mgh; \quad V_C^2 = \frac{2gh}{\left(1+\dfrac{K^2}{R^2}\right)}$$

Thus the velocity of center of mass of a body rolling down an inclined plane is given by

$$V_C = \frac{\sqrt{2gh}}{\left(1 + \frac{K^2}{R^2}\right)^{1/2}}$$

If a_C is linear acceleration of center of mass down this plane, and distance covered on the plane is s, then if the body starts from rest we have

$$V_C^2 = 2a_C s \therefore a_C = \frac{V_C^2}{2\,s} = \frac{2\,gh}{\left(1 + \frac{K^2}{R^2}\right) \times 2 \times \frac{h}{\sin\theta}} \text{ or } a_C = \frac{g \sin\theta}{1 + \frac{K^2}{R^2}}$$

MASTERJEE CONCEPTS

Rather than going in a conventional way, using this method greatly simplifies our effort. But take care while writing the kinetic energy!

Nitin Chandrol (JEE 2012 AIR 134)

Illustration 37: A solid sphere is released from rest from the top of an incline of inclination θ and length ℓ. If the sphere rolls without slipping. What will be its speed when it reaches the bottom? **(JEE MAIN)**

Sol: The loss in the gravitational potential energy of the solid sphere is equal to the gain in the kinetic energy. The kinetic energy of the sphere comprises the rotational kinetic energy as well as the translational kinetic energy.

Let the mass of the sphere be m and its radius be r. Suppose the linear speed of the sphere when it reaches the bottom is v. As the sphere rolls without slipping, its angular speed ω about its axis is v/r. The kinetic energy at the bottom will be

$$K = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 = \frac{1}{2}\left(\frac{2}{5}mr^2\right)\omega^2 + \frac{1}{2}mv^2 = \frac{1}{5}mv^2 + \frac{1}{2}mv^2 = \frac{7}{10}mv^2$$

This should be equal to the loss of potential energy mg ℓ sinθ. Thus

$$\frac{7}{10}mv^2 = mg\ell\sin\theta \quad \text{Or} \quad v = \sqrt{\frac{10}{7}g\ell\sin\theta}.$$

11. TOPPLING

When an external force is applied to the upper edge of a body with a flat base to cause it to slide along a surface, the body may topple before sliding starts. Toppling is more likely to happen when the width of the base of the body is small.

Toppling occurs due to the turning effect of torques of applied force at the upper edge and frictional force at the base.

Let the surface be quite rough and the force F is applied at height h above the base of the block as shown in Fig. 7.61. Width of the base is b. The static friction at the base is f = F. The normal reaction is N = mg. The couple of forces F and f try to topple the block about point S. To cancel the effect of this unbalanced torque the normal reaction N shifts towards S by a distance x so that torque of N counter balances torques of F and f.

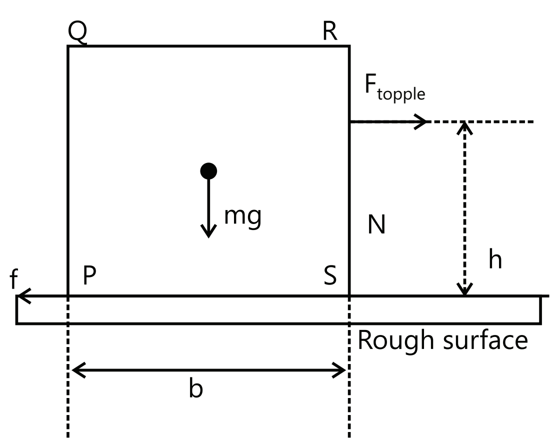


Figure 7.61: Block toppling on rough surface

$Fh = (mg)x \quad \text{or} \quad x = \dfrac{Fh}{mg}$

If F or h or both increase, distance x also increases, but it cannot go beyond the maximum value of x_{max} = b/2 i.e in extreme case N passes through edge S. If F is further increased block will topple.

So, $F_{topple} = \dfrac{mgb}{2h}$

Here we assumed that the surface is sufficiently rough so that sliding starts only when

$F = f_{max} = \mu mg > F_{topple}$ or $\mu > \dfrac{b}{2h}$ (toppling before sliding)

If surface is not sufficiently rough, the body slides before F is increased to F_{topple} i.e. the body will slide before toppling. This is the case when

$F = f_{max} = \mu mg < F_{topple}$ or $\mu < \dfrac{b}{2h}$

Illustration 38: A uniform cube of side 'a' and mass m rests on a rough horizontal table. A horizontal force F is applied normal to one of the faces at a point directly below the center of the face, at a height $\frac{a}{4}$ above the base.

(i) What is the minimum value of F for which the cube begins to tip about an edge?

(ii) What is the minimum value of μ_s so that toppling occurs?

(iii) If $f_1 = \mu_{min}$, find minimum force for toppling.

(iv) Minimum μ_s so that F_{min} can cause toppling. **(JEE ADVANCED)**

Sol: For part (i) we consider toppling before sliding. The normal reaction will pass through the edge. In part (ii) it is not mentioned whether the toppling occurs before sliding or sliding occurs before toppling. So the toppling will occur for any value of μ_s, sliding or no sliding. Part (iii) is same as part (i). Part (iv) is the case of toppling before sliding.

(i) In the limiting case normal reaction will pass through O. The cube will tip about O if torque of F about O exceeds the torque of mg.

Hence, $F\left(\dfrac{a}{4}\right) > mg\left(\dfrac{a}{2}\right)$ or $F > 2\,mg$

Therefore, minimum value of F is 2 mg.

(ii) In this case since it is not acting at COM, toppling can occur even after body started sliding even if there is no friction by increasing the torque of F about COM. Hence $\mu_{min} = 0$.

(iii) Now body is sliding before toppling. O is not I.A.R., torque equation cannot be applied across it. It can be applied about COM.

$F \times \dfrac{a}{4} = N \times \dfrac{a}{2}$ ... (i)

$N = mg$ ... (ii)

From (i) and (ii) -> F = 2 mg

(iv) F > 2 mg .... (i) (From sol. (i))

$N = mg$ ... (ii)

$F = \mu_s N = \mu_s mg$ ... (iii)

From (i) and (iii) $\mu_s = 2$

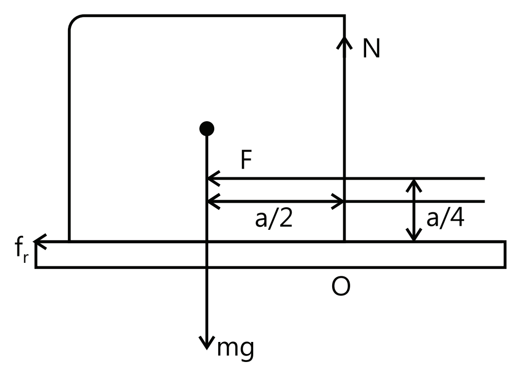


Figure 7.62

Illustration 39: Find minimum value of ℓ so that truck can avoid the dead end, without toppling the block kept on it. **(JEE ADVANCED)**

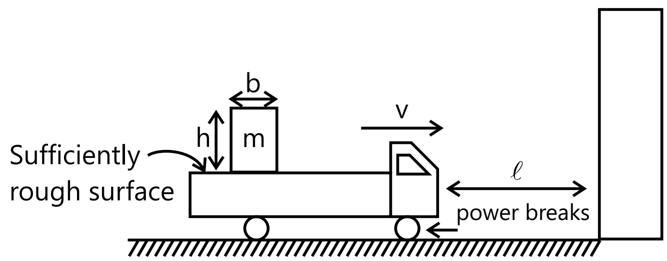


Figure 7.63

Sol: The block kept on truck will experience pseudo force in forward direction and friction force due to the floor of the truck in backward direction. We assume the case of toppling before sliding. In extreme case the normal reaction N = mg will pass through the edge.

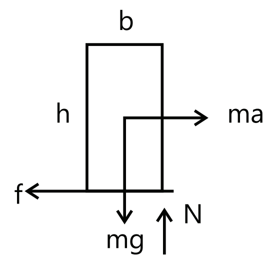


Figure 7.64

$$ma\frac{h}{2} \le mg\frac{b}{2} \Rightarrow a \le \frac{b}{h}g$$

Final velocity of truck is zero. So that $0 = v^2 - 2(\frac{b}{h}g)\ell$

$$\ell = \frac{h}{2b}\frac{v^2}{g}$$

PROBLEM-SOLVING TACTICS

- Most of the problems involving incline and a rigid body can be solved by using conservation of energy during pure rolling. In case of non-conservative forces, work done by them also has to be taken into consideration in the equation. Care has to be taken in writing down the Kinetic energy. Rotational Kinetic Energy term has to be taken into consideration. And while writing the rotational energy, the axis about which the moment of inertia is taken should pass through the COM.
- The motion of a body in pure rolling can be viewed as pure rotation about the bottommost point of the body or the point of contact with the ground. Hence an axis passing through the point of contact and tangential to the point would be the Instantaneous axis of rotation. So problems on pure rolling can be solved easily by using the concept of instantaneous axis of rotation.
- Problems on toppling can be easily solved by writing the moments on the body and visualizing them as forces acting on the body. If the net moment is tending to stabilize the body, then the body doesn't topple. For any condition else it may get toppled.
- Problems which include the concept of sliding and rolling can be solved easily by using the concept of conservation of angular momentum. But care has to be taken in selecting the proper axis so that net moment about that axis vanishes.

FORMULAE SHEET

S. No	Term	Description	Linear Motion	Rotational motion & relation
1	Displacement	Displacement (linear or angular) is the physical change in the position of the body when a body moves linearly or angular in position. **(a)** The linear displacement Δs is difference between final and initial position measured in linear direction. **S.I. unit**: meter **m** **(b)** The angular displacement of the body while rotating about a fixed axis is the displacement $\Delta\theta$ it swept out with respect to its initial position in sense of rotation. It can be positive (anti clockwise) or negative (clockwise) **S.I. unit:** radians **rad**,	s	θ $(s = r\theta)$
2	Velocity	Velocity of any moving object is the time rate of change of position. The velocity is the vector quantity. Linear velocity is in the plane of motion. Angular velocity can be positive or negative & its direction is perpendicular to the plane of rotation Linear velocity is categorized as - Average velocity= $\Delta s / \Delta t$ - Instantaneous velocity= ds/dt. **S.I. unit: m/s** Angular velocity is categorized as - Average angular velocity $\Delta\theta / \Delta t$ - Instantaneous angular velocity $\omega = d\theta / dt$ **S.I. unit: rad/s**	$v = \frac{ds}{dt}$	$\omega = \frac{d\theta}{dt}$ $(v = r\omega)$
3	Acceleration	Acceleration is the time rate change of velocity of a body. It's a vector quantity. Linear acceleration can be positive or negative and related to direction of motion. Linear acceleration is categorized as - Average acceleration= $\Delta v / \Delta t$ - Instantaneous acceleration = dv/dt.	$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$ $(a = r\alpha)$

S. No	Term	Description	Linear Motion	Rotational motion & relation
		S.I. unit: m/s⁻² Angular acceleration is categorized as - Average angular acceleration $\Delta\omega / \Delta t$ - Instantaneous angular acceleration $\alpha = d\omega / dt$ **S.I. unit: rad/s⁻²**		
4	Mass	Mass is the basic entity of any body by virtue of which the body gains weight. In linear kinematics the mass of whole body is constant. **S.I. unit:** kilogram **kg** In angular kinematics mass of body is distributed among various tiny rigid points so mass is measured about inertia of rotating body- moment of inertia **I**	M	I ($I = \sum mr^2$)
5	Momentum	Momentum of body is product of mass and its velocity of motion. It's a vector quantity. Linear momentum= mv **S.I. unit: kg m/s** Angular momentum of body is a vector in direction perpendicular to plane of rotation given by $\vec{L}$ **S.I. unit: kg m²/s**	p = mv	$\vec{L}$ = I $\vec{L} = \vec{r} \times \vec{p}$
6	Impulse	Impulse is the product of force and time period And it is categorized as -Linear impulse -Angular impulse	$\int F\, dt$	$\int \tau dt$
7	Force (Newton's second law of motion)	From the newton second law of motion, force is time rate of change of momentum. It's a vector quantity. Linear force F= $\frac{dp}{dt}$ = ma **S.I. unit:** Newton **N** Angular force $\vec{\tau} = I \times \vec{\alpha}$ **Laws of conservation of momentum** - Linear momentum is said to be conserved if $\frac{dp}{dt}$ = 0, than P remains constant - Angular momentum is said to be conserved if $\frac{d\vec{L}}{dt}$ = 0 than L remains constant	F = ma If = 0 the body is in equilibrium with its surrounding	$\vec{\tau} = r \times \vec{F} = I \times \vec{\alpha}$ $= \frac{d\vec{L}}{dt}$ If = 0 the body is in equilibrium with its surrounding

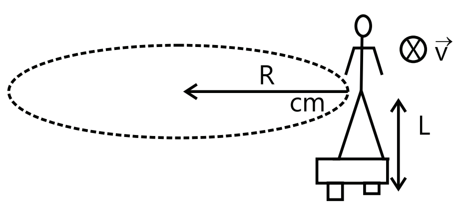
S. No	Term	Description	Linear Motion	Rotational motion & relation
8	Work	Work is the product of displacement of body under action of external applied force.	$W = \int F\,ds$	$W = \int \tau d\theta$
9	Power	Power is the time rate change of work done	P =F	$P = \tau\,\omega$
10	Kinetic energy	The phenomenon associated with the moving bodies	$K.E._{tran} = \frac{1}{2}mv^2$	$K.E._{rot} = \frac{1}{2}I\omega^2$
11	Kinematics of Motion	Kinematical equation are the interrelation of displacement, velocity, acceleration and time and are categorized as follows: -Linear kinematical equation -Angular kinematical equation	$v = u + at$ $s = ut + \frac{1}{2}at^2$ $v^2 = u^2 + 2as$	$\omega = \omega_0 + \alpha t$ $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ $\omega^2 = \omega_0^2 + 2\alpha\theta$
12	Parallel Axis Theorem	$I_{xx} = I_{cc} + Md^2$ where I_{cc} is the moment of inertia about the center of mass		
13	Perpendicular Axis Theorem	$I_{xx} + I_{yy} = I_{zz}$ It is valid for plane laminas only.		
14	Work energy principle	Work energy principle is used to determine the change in the kinetic energy of moving body	$W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$	$W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$

Solved Examples

JEE Main/Boards

The first five Examples discussed below show us the strategy to tackle down any problem in the rigid body motion. Hence follow them up properly! They may be lengthy but are very learner friendly!!

Example 1: A person of mass M is standing on a railroad car, which is rounding an unbanked turn of radius at speed v. His center of mass is at a height of L above the car midway between his feet, which are separated by a distance of d. The man is facing the direction of motion. What is the magnitude of the normal force on each foot?



Sol: The frictional forces acting on the feet of man will provide the necessary centripetal acceleration to move in a circular path. Apply the Newton's second law of motion at the center of mass of the man to get the equation of motion along the circular path. In the vertical plane the man is in rotational and translational equilibrium under the action of its weight acting vertically downwards and the normal reactions at its feet acting vertically upwards. Get one equation each