# 3. MOTION IN A PLANE

## 1. INTRODUCTION

Motion in a plane is a two dimensional motion. The analysis of this type of motion becomes easy when we consider this motion as a combination of two straight line motions along two mutually perpendicular axes lying in the plane of motion. In Cartesian coordinate system the two mutually perpendicular axes are the x-axis and the y-axis respectively. The displacement, velocity and acceleration of the particle are resolved into components along the x and y axes and motion along each axis is studied independent of the other. The net displacement, velocity or acceleration is the vector sum of their respective components along the two axes. In this chapter we will discuss about the motion of a projectile, the motion of a body relative to another body, the motion of a body in a river, the motion of an airplane with respect to wind, and circular motion.

## 2. MOTION IN A PLANE

When a body moves in a straight line, we call it motion in a straight line or one dimension. For eg, a car going straight on a road. When you throw a ball towards your friend, the ball follows a non-linear path. This motion is termed as motion in two dimensions or motion in a plane.

The position of a particle that is free to move can be located by two coordinates in a plane. We choose the plane of motion as the X-Y plane. We choose a suitable instant as t = 0 and choose the origin at the place where the particle is situated at t = 0. Any two convenient mutually perpendicular directions in the X-Y plane are chosen as the X and Y-axes.

## 3. PROJECTILE MOTION

Projectile motion is a form of motion in which an object or particle (here called a projectile) is thrown in an oblique direction near the earth's surface, and it moves along a curved path under the action of a continuous motive force. The path observed during a projectile motion is called its trajectory. Projectile motion is possible only when there is one force applied at the beginning of the trajectory, after which there is no force in operation except a constant force.

### 3.1 Ground-To-Ground Projectile

In the Fig. 3.2 shown, let us consider the horizontal surface through the point O. Now, the point O here is called the point of projection, the angle $\theta$ is called the angle of projection and the distance OB is called the horizontal range or simply range. Further, the total time taken by the particle in describing the path OAB is called the time of flight.

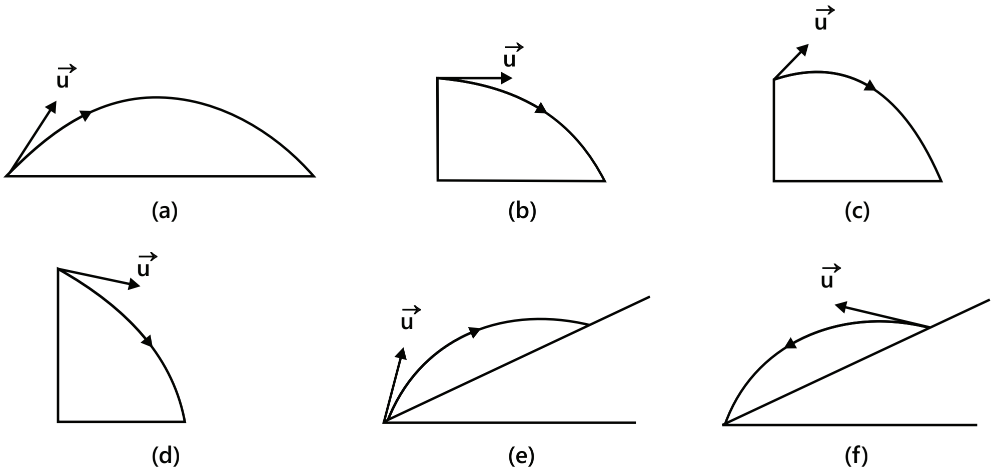



**Figure 3.1**

However, we can separately discuss the motion of the projectile for both the horizontal and vertical parts. In this regard, we begin by considering the origin as the point of projection.

Now, we have $u_x = u\cos\theta;\ a_x = 0;\qquad u_y = u\sin\theta;\ a_y = -g.$

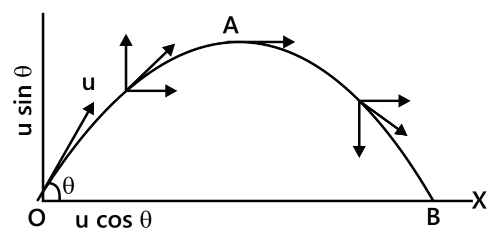



**Figure 3.2**

### 3.1.1 Horizontal Motion

As $\alpha_x = 0$, we have $\nu_x = u_x + \alpha_x t = u_x = u\cos\theta$ and $x = u_x t + \frac{1}{2}\alpha_x t^2 = u_x t = ut\cos\theta$

### 3.1.2 Vertical Motion

In the downward direction, we know that the acceleration of the particle is g. Thus, $\alpha_y = -g$.

Further, the y-component of the initial velocity is $u_y$. Thus,

$\nu_y = u_y - gt$ and $y = u_y t - \frac{1}{2}gt^2$; also we have, $\nu_y^2 = u_y^2 - 2gy$.

### 3.1.3 Time of Flight

Let us suppose that the particle is at B at time t. Therefore, the equation for horizontal motion gives OB = x = ut $\cos\theta$.

However, the y-coordinate at the point B is zero. Thus, from the equation of vertical motion,

$$y = ut\sin\theta - \frac{1}{2}gt^2 \quad \text{or,} \quad 0 = ut\sin\theta - \frac{1}{2}gt^2 \quad \text{or,} \quad t(u\sin\theta - \frac{1}{2}gt) = 0$$

Thus, either t = 0 or, $t = \dfrac{2u\sin\theta}{g}$

Now, t = 0 exactly corresponds to the initial position O of the particle. Hence, the time at which it reaches B is thus,

$$T = \frac{2u\sin\theta}{g}$$

This equation helps us to exactly calculate the time of flight.

### 3.1.4 Range

Consider the distance OB covered by a particle, which is the horizontal range. It is the distance travelled by the particle in time $T = \frac{2u\sin\theta}{g}$

By the equation of horizontal motion, $x = (u\cos\theta)\times T$ or, $OB = \frac{2u^2\sin\theta\cos\theta}{g} = \frac{u^2\sin 2\theta}{g}$

### 3.1.5 Maximum Height

We have, $v_y = u_y - gt = u\sin\theta - gt$

However, at the maximum height, $0 = u\sin\theta - gt$ or, $t = \frac{u\sin\theta}{g}$

The actual maximum height is $H = u_y t - \frac{1}{2}gt^2 = (u\sin\theta)\left(\frac{u\sin\theta}{g}\right) - \frac{1}{2}g\left(\frac{u\sin\theta}{g}\right)^2$

$$= \frac{u^2\sin^2\theta}{g} - \frac{1}{2}\frac{u^2\sin^2\theta}{g} = \frac{u^2\sin^2\theta}{2g}$$

## 4. EQUATION OF TRAJECTORY OF A PROJECTILE

$x = (u\cos\alpha)t \qquad \therefore\ t = \frac{x}{u\cos\alpha}$

By substituting this value of t in $y = (u\sin\alpha)t - \frac{1}{2}gt^2$, we obtain

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha = x\tan\alpha - \frac{gx^2}{2u^2}(1+\tan^2\alpha)$$

The above are the standard equations of trajectory of any projectile. Here, we should be aware of the fact that the equation is quadratic in x. This is why the path of a projectile is always a parabola. Further, the above equation can also be represented in terms of range (R) of the projectile as $y = x\left(1 - \frac{x}{R}\right)\tan\alpha$

**Illustration 1:** Assume that a ball is thrown from a field at a speed of 12.0 m/s and at an angle of 45° with the horizontal. At what distance will it hit the field again? Take g = 10.0 m/$s^2$. **(JEE MAIN)**

**Sol:** Use the formula for the range of a projectile.

The horizontal range $= \frac{u^2\sin 2\theta}{g} = \frac{(12m/s)^2\times\sin(2\times 45^\circ)}{10m/s^2} = \frac{144m^2/s^2}{10.0m/s^2} = 14.4m$

Thus, the ball hits the field exactly at 14.4 m from the point of projection.

## MASTERJEE CONCEPTS

(i) Range is maximum where $2\alpha = 1$ or $\alpha = 45^\circ$ and this maximum range is:

$$R_{max} = \frac{u^2}{g} = 4H$$

(ii) For given value of u, range at $\alpha$ and range at $\phi$ are equal although times of flight and maximum heights may be different. Because

$$R_{90^o-\alpha} = \frac{u^2 \sin 2(90^o - \alpha)}{g} = \frac{u^2 \sin(180^o - 2\alpha)}{g} = \frac{u^2 \sin 2\alpha}{g} = R_\alpha$$

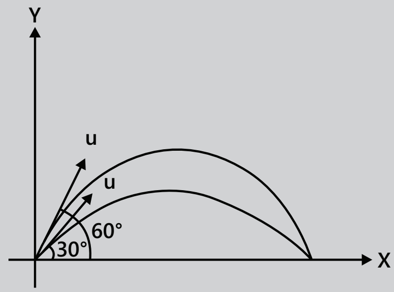


**Figure 3.3**

As we have seen in the above derivations that $a_x = 0$, i.e., motion of the projectile in the horizontal direction is uniform. Hence, horizontal component of velocity $u \cos\alpha$ does not change during its motion. Motion in the vertical direction is first retarded and then accelerated in opposite direction. As the equation of trajectory of projectile is of the form, $y = ax - bx^2$ (equation of parabola), therefore, the path followed by a projectile is a parabola.

**B Rajiv Reddy (JEE 2012, AIR 11)**

**Illustration 2:** Find the angle of a projectile for which both the horizontal range and maximum height are equal. **(JEE MAIN)**

**Sol:** Use the formula for the range and maximum height of a projectile.

Given, R = H

$$\therefore \quad \frac{u^2 \sin 2\alpha}{g} = \frac{u^2 \sin^2\alpha}{2g} \quad \text{or} \quad 2\sin\alpha\cos\alpha = \frac{\sin^2\alpha}{2} \quad \text{or} \quad \frac{\sin\alpha}{\cos\alpha} = 4 \text{ or } \tan\alpha = 4 \quad \alpha = \tan^{-1}(4)$$

**Illustration 3:** The given Fig. 3.4 shows a pirate ship 560 m from a fort defending a harbor entrance. A defense canon, located at sea level, fires balls at initial speed $v_o = 82 m/s$ **(JEE ADVANCED)**

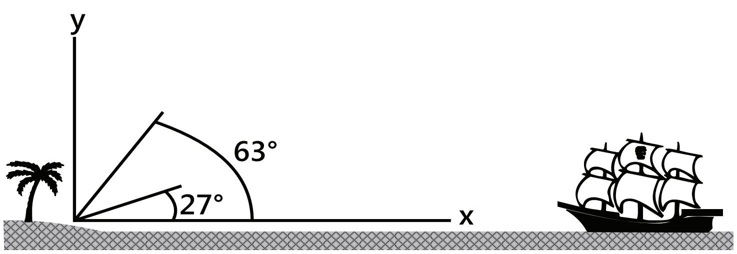


**Figure 3.4**

(a) At what angle $\theta_o$ from the horizontal must a ball be fired to hit the ship?

**Sol:** Use the formula for the range of a projectile to find the angle of projection.

We can relate the launch angle $\theta_o$ to the range R with Eq. $(R = (v_0^2 / g)\sin 2\theta_o)$, which, after rearrangement, gives

$$\theta_o = \frac{1}{2}\sin^{-1}\frac{gR}{v_0^2} = \frac{1}{2}\sin^{-1}\frac{(9.8 m/s^2)(560 m)}{(82 m/s)^2} = \frac{1}{2}\sin^{-1} 0.816$$

One solution of $(54.7^\circ)$ is worked out using a calculator; now, we subtract it from $180^\circ$ to get the other solution $(125.3^\circ)$. This gives us $\theta_0 = 27^\circ$ and $\theta_0 = 63^\circ$.

**Illustration 4:** Suppose a batsman B hits a high-fly ball to the outfield, directly toward an outfielder F and with a launch speed of $v_\circ = 40\,m/s$ and a launch angle of $\theta_0 = 35^\circ$. During the flight, a line from the outfielder to the ball makes an angle $\phi$ with the ground. Based on the data provided, plot the elevation angle $\tan\theta = 2\cot\alpha$ versus time t, assuming that the outfielder is (a) already positioned to catch the ball, (b) is 6.0 m too close to the batsman and (c) is 6.0 m too far away. **(JEE ADVANCED)**

**Sol:** While trying to catch a ball which has gone to a great height you can imagine that the angle of line of sight increases as the ball moves. If we neglect air drag, then the ball is a projectile for which the vertical motion and the horizontal motion can be analyzed individually.

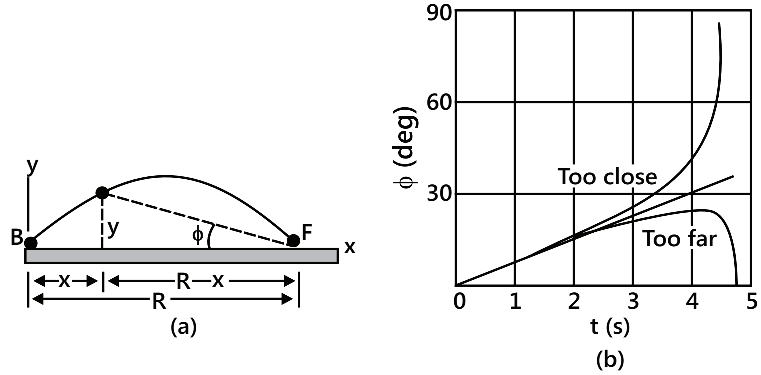



**Figure 3.5**

Assuming that the ball is caught at approximately the height it is hit, the horizontal distance traveled by the ball is the range R, given by Eq. $(R = (v_0^2/g)\sin 2\theta_\circ)$

The ball can be caught if the outfielder's distance from the batsman equals the range R of the ball. Using the above equation, we find the elevation angle $\phi$ for a ball that was hit toward an outfielder is (a) defined and (b) plotted versus time t.

$$R = \frac{v_0^2}{g}\sin 2\theta_\circ = \frac{(40\,m/s)^2}{9.8\,m/s^2}\sin(70^\circ) = 153.42\,m$$

Fig. 3.5 (a) above shows a snapshot of the ball in flight when the ball is at height y and horizontal distance x from the batsman (who is at the origin). The horizontal distance of the ball from the outfielder is R – x, and the elevation angle $\phi$ of the ball in the outfielder's view is given by $\tan\phi = y/(R-x)$.

Thus, using $v_0 = 40$ m/s and $\theta_0 = 35^\circ$, we have $\phi = \tan^{-1}\dfrac{(40\sin 35^\circ)t - 4.9t^2}{153.42 - (40\cos 35^\circ)t}$

By graphing this function versus t gives us the middle plot in b. We now see that the ball's angle in the outfielder's view increases at an almost steady rate throughout the flight.

If the outfielder is 6.0 m close to the batsman, then we replace the distance of 153.42 m in the given equation with 153.42 m – 6.0 m = 147.42 m. Further, regraphing the function gives the "Too close" plot as in Fig. 3.5 (b).

Now, we observe that the elevation angle of the ball rapidly increases toward the end of the flight as the ball soars over the outfielder's head. However, if the outfielder is 6.0 m too far away from the batsman then we replace the distance of 153.42 m in the equation with 159.42 m. The resulting plot is hence labeled "Too far" in the Fig. 3.5: The angle first increases and thereafter rapidly decreases.

Conclude: Thus, if a ball is hit directly toward an outfielder, then the player can tell from the change in the ball's elevation angle $\phi$ whether to stay put, run toward the batter, or back away from the batsman.

**Illustration 5:** Suppose that a projectile is fired horizontally with a velocity of 98 m/s from the top of a hill that is 490 m high. Find:

(a) The time taken by the projectile to reach the ground,

(b) The distance of the point where the particle hits the ground from the foot of the hill and

(c) The velocity with which the projectile hits the ground. (take $g = 9.8\,m/s^2$) **(JEE MAIN)**

**Sol:** Let x-axis be along the horizontal and the y-axis be along the vertical. The projectile will have uniform velocity along the positive x-axis and uniform acceleration along the negative y-axis.

In this problem, we cannot apply the formulae of R, H and T directly. Necessarily we have to follow the three steps discussed in the theory. Here, however, it will be more convenient to choose x and y directions as shown in the Fig. 3.6 provided.

Here, $u_x = 98\,m/s, a_x = 0, u_y = 0$ and $a_y = g$

(a) At A, $S_y = 490\,m$. Therefore, applying $s_y = u_y t + \frac{1}{2}a_y t^2$

$\therefore \quad 490 = 0 + \frac{1}{2}(9.8)t^2 \quad \therefore \quad t = 10s$

(b) $BA = s_x = u_x t + \frac{1}{2}a_x t^2$ or $BA = (98)(10) + 0$ or $BA = 980\,m$

(c) $v_x = u_x = 980 m/s; \quad v_y = u_y + a_y t = 0 + (9.8)(10) = 98 m/s$

$\therefore \quad v = \sqrt{v_x^2 + v_y^2} = \sqrt{(98)^2 + (98)^2} = 98\sqrt{2}\,m/s$

and $\tan\beta = \frac{v_y}{v_x} = \frac{98}{98} = 1 \quad \therefore \quad \beta = 45^\circ$

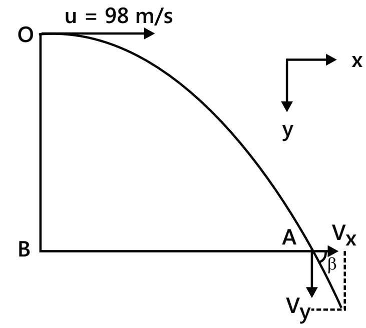


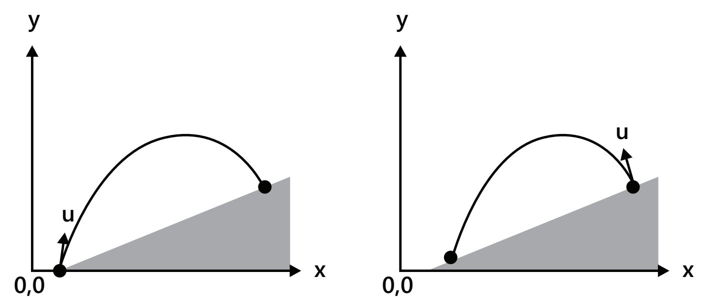
Figure 3.6

Thus, we show that the projectile hits the ground with a velocity $98\sqrt{2}\,m/s$ at an angle of $\beta = 45^\circ$ with horizontal as shown in the Fig. 3.6 provided.

## 5. PROJECTILE MOTION ON AN INCLINED PLANE

Projectile motion on an incline plane is one of the various types of projectile motion. However, the main distinguishing aspect is that points of projection and return are not on the same horizontal plane.

We know that there are two possibilities in this regard: (i) the point of return is at a higher level than the point of projection i.e., projectile is thrown up the incline and (ii) the point of return is at a lower level than the point of projection, i.e., the projectile is thrown down the inclined plane.



**(a) Projection up the incline** **(b) Projection down the incline**

Figure 3.7

### 5.1 Analyzing Motion

We can make us of two different approaches of analyzing projectile motion on an inclined plane. The first approach preferably could be to continue analyzing motion in two mutually perpendicular horizontal and vertical directions. The second approach, therefore, could be to analyze motion by changing the reference orientation, i.e., we set up our coordinate system along the incline and a direction along the perpendicular to the incline.

Based on the analysis, alternatives are, therefore, distinguished on the basis of coordinate system that we choose to employ:

**(a)** Planar coordinates along the incline (x) and perpendicular to the incline (y)

**(b)** Planar coordinates in horizontal (x) and vertical (y) directions

However, we use the first approach for analyzing this kind of motion, i.e., coordinates along the incline (x) and perpendicular to the incline (y).

## 5.2 Projectile Motion Up an Inclined Plane

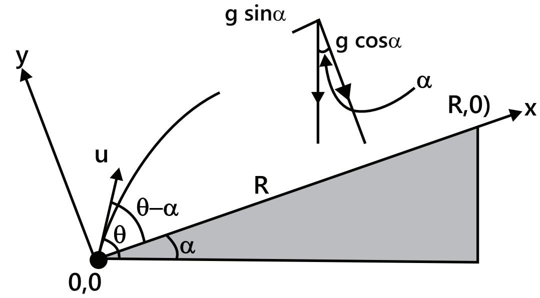



Figure 3.8

Based on the details provided in the Fig. 3.8, it is clear that the angle that the velocity of projection makes with the x-axis (i.e., incline) is " $\theta - a$ ".

Therefore, the components of initial velocity are

$u_x = u\cos(\theta-\alpha);\quad u_y = u\sin(\theta-\alpha)$

Hence, the components of acceleration are

$a_x = -g\sin\alpha;\quad a_y = -g\cos\alpha$

### 5.2.1 Time of Flight

The time of flight (T) is calculated by analyzing motion in y-direction (which is not vertical as in the normal case). However, the displacement in the y-direction after the projectile has returned to the incline is zero as in the normal case. Thus,

$$y = u_y T + \frac{1}{2}a_y T^2 = 0 \Rightarrow u\sin(\theta-\alpha)T + \frac{1}{2}(-g\cos\alpha)T^2 = 0 \quad \Rightarrow T\left\{u\sin(\theta-\alpha)T + \frac{1}{2}(-g\cos\alpha)T\right\} = 0;\ T = 0$$

$$\Rightarrow T = \frac{2u\sin(\theta-\alpha)}{g\cos\alpha}$$

Here, the first value represents the initial time of projection. Hence, the second expression gives us the time of flight. However, we should note here that the expression of time of flight is as in a normal case albeit in a significant manner.

In the generic form, we can express the formula of the time of flight as: $T = \left|\frac{2u_y}{a_y}\right|$

### 5.2.2 Range of Flight

The first thing that we should note that we do not use the term "horizontal range" as the range on the inclined plane is no more horizontal. Rather, we simply refer the displacement along the x-axis as "range". Thus, we can find range of flight by considering motion in both "x" and "y" directions. We further note that we utilize the same approach even in the normal case. Now, let "R" be the range of the projectile motion.

Substituting the value of "T" as obtained earlier, we have $x = u_x T - \frac{1}{2}a_x T^2$

$$R = \frac{u\cos(\theta-\alpha)\times 2u\sin(\theta-\alpha)}{g\cos\alpha} - \frac{g\sin\alpha\times 4u^2\sin^2(\theta-\alpha)}{2g^2\cos^2\alpha}$$

$$\Rightarrow R = \frac{u^2}{g\cos^2\alpha}\left\{2\cos(\theta-\alpha)\sin(\theta-\alpha)\cos\alpha - \sin\alpha\times 2\sin^2(\theta-\alpha)\right\}$$

Using trigonometric relation, $2\sin^2(\theta-\alpha) = 1-\cos 2(\theta-\alpha)$,

$$\Rightarrow R = \frac{u^2}{g\cos^2\alpha}\left[\sin 2(\theta-\alpha)\cos\alpha - \sin\alpha\left\{1-\cos 2(\theta-\alpha)\right\}\right]$$

$$\Rightarrow R = \frac{u^2}{g\cos^2\alpha}\left\{\sin 2(\theta-\alpha)\cos\alpha - \sin\alpha + \sin\alpha\cos 2(\theta-\alpha)\right\}$$

Now, we use the trigonometric relation, $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$\Rightarrow R = \frac{u^2}{g\cos^2\alpha}\left\{\sin(2\theta-2\alpha+\alpha) - \sin\alpha\right\} \Rightarrow R = \frac{u^2}{g\cos^2\alpha}\left\{\sin(2\theta-\alpha) - \sin\alpha\right\}$$

This is the exact expression for the range of projectile on an inclined plane. We also note that this expression reduces to the one for the normal case, when $\alpha = 0 \quad \Rightarrow R = \frac{u^2 \sin 2\theta}{g}$

## 5.3 Projectile Motion down the Inclined Plane

The components of initial velocity: $u_x = u\cos(\theta+\alpha);\ u_y = u\sin(\theta+\alpha)$

The components of acceleration: $a_x = g\sin\alpha;\ a_y = -g\cos\alpha$

Time of flight

The expression for the time of flight differs only with respect to angle of sine function in the numerator of the expression: $T = \frac{2u\sin(\theta+\alpha)}{g\cos\alpha}$

Range of flight

In the same way, the expression of range of flight differs only with respect to angle of sine function: $R = \frac{u^2}{g\cos^2\alpha}\{\sin(2\theta+\alpha)+\sin\alpha\}$

### MASTERJEE CONCEPTS

It is very handy to note that expressions have changed only with respect of the sign of "$\alpha$" for the time of flight and the range. We only need to exchange "$\alpha$" by"$-\alpha$ s".

**Vaibhav Krishan (JEE 2009, AIR 22)**

**Illustration 6:** Assume that a projectile is thrown from the base of an incline of angle 30° as shown in the Fig. 3.9 provided. It is thrown at an angle of 60° from the horizontal direction at a speed of 10 m/s. Calculate the total time of flight is (consider g = 10 m/s²). **(JEE MAIN)**

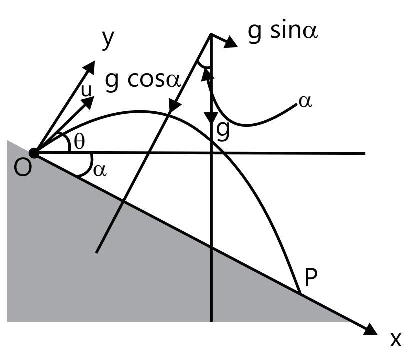


**Figure 3.9**

**Sol:** The x-axis has to be assumed along the inclined.

This problem can be handled with a reoriented coordinate system as shown in the Fig. 3.9 provided. Here, the angle of projection with respect to x-direction is ($\theta -$ a) and acceleration in y-direction is "g cosa". Now, the total time of flight for the projectile motion, when the point of projection and return are on the same level, is

$$\Rightarrow T = \frac{2u\sin(\theta - a)}{g\cos\alpha}$$

Now, $\theta = 60^\circ, a = 30^\circ$, u = 10m/s. Then, by substituting these values, we finally obtain

$$\Rightarrow T = \frac{2X10\sin(60^\circ - 30^\circ)}{g\cos 30^\circ} = \frac{20\sin 30^\circ}{10\cos 30^\circ} = \frac{2}{\sqrt{3}}$$

**Illustration 7:** Consider that two projectiles are thrown with the same speed from point "O" and "A" so that they hit the incline. If $t_0 \text{and} t_A$ be the time of flight in two cases, then prove which option out of those given here is true. **(JEE MAIN)**

(A) $t_0 = t_A$ (B) $t_0$ & $t_A$ (C) $t_0 > t_A$ (D) $t_0 = t_A = \frac{u\tan\theta}{g}$

**Sol:** The x-axis has to be assumed along the inclined. Use the formula for time of flight on an inclined. Let us first consider the projectile thrown from the point "O". Considering the angle the velocity vector makes with the horizontal, we represent the time of flight as:

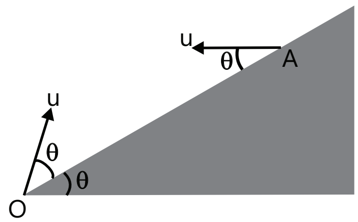



Figure 3.10

$$\Rightarrow t_0 = \frac{2u\sin(2\theta-\theta)}{g\cos\theta} \quad \Rightarrow t_0 = \frac{2u\tan\theta}{g}$$

Further, for the projectile thrown from the point "A", the angle with horizontal is zero. Hence, the time of flight is expressed as

$$\Rightarrow t_A = \frac{2u\sin(2X0-\theta)}{g\cos\theta} = \frac{2u\tan\theta}{g}$$

Thus, we observe that the times of flight in the two cases are equal.

$$\Rightarrow t_A = t_0$$

Hence, option (A) is correct.

**Illustration 8:** Two inclined planes of angles 30° and 60° are placed so that they touch each other at the base as shown in the Fig. 3.11 provided. Further, a projectile is projected at right angle at a speed of $10\sqrt{3}$ m/s from point "P" and hits the other incline at point "Q" normally. Then, the time of flight is: **(JEE ADVANCED)**

(A) 1 s (B) 2 s (C) 3 s (D) 4s

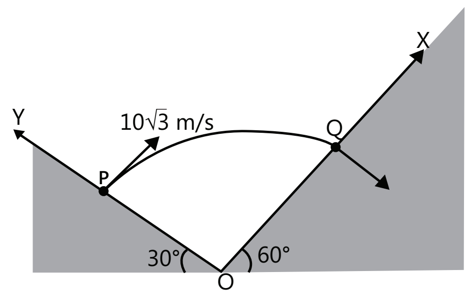



Figure 3.11

**Sol:** This problem is a specific case in which the inclined planes are right angles with respect to each other. Therefore, we actually take advantage of this fact in assigning our coordinates along the planes, say y-axis along first incline and x-axis against second incline.

Thus, in order to find the time of flight, we can further use the fact that projectile hits the other plane at right angle, i.e., parallel to the y-axis. This means that the component of velocity in x-direction, i.e., along the second incline is zero. This, in turn, suggests that we can analyze motion in x-direction to obtain the time of flight.

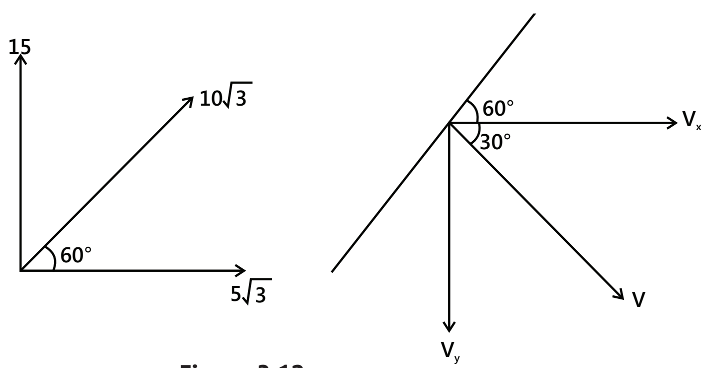



Figure 3.12

$V_x = 10\sqrt{3}\cos 60 = 5\sqrt{3}$ will remain constant

$\therefore$ Here, $V\cos 60 = 5\sqrt{3}$

$\therefore$ $V = \frac{5\sqrt{3}}{\sqrt{3}} X2 = 10 \quad \therefore V_y = -V\sin 30 = \frac{-V}{2} = -5$

$\therefore$ $V = u - gt \qquad \therefore -5 = 15 - 10t$

$\therefore$ $10t = 20 \qquad \therefore \boxed{t = 2s}$

## MASTERJEE CONCEPTS

Given are a simple set of guidelines in a very general way:

- Analyze motion independently along the selected coordinates for complicated problems. For simple cases, try remembering derived formula and use them directly to save time.
- Make note of information given in the question like angles, etc., which might render certain components of velocity zero in certain direction.
- If range of the projectile is given, we may try the trigonometric ratio of the incline itself to get the answer.
- If we use coordinate system along incline and in the direction perpendicular to it, then always remember that component of motion along both incline and in the direction perpendicular to it are accelerated motions. Ensure that we use appropriate components of acceleration in the equations of motion.
- The range is maximum for maximum value of "sin(2θ–α)". Thus, the range is maximum for the angle of projection as measured from horizontal direction, when

$$\sin(2\theta-\alpha)=1 \Rightarrow \sin(2\theta-\alpha)=\sin\pi/2$$

$$\Rightarrow\ 2\theta-\alpha=\pi/2 \ \Rightarrow\ \theta=\pi/4+\alpha/2 \Rightarrow\ R_{max}=\frac{u^2}{g\cos^2\alpha}(1-\sin\alpha)$$

**Anand K (JEE 2011, AIR 47)**

## 6. RELATIVE MOTION

The measurements describing motion are generally subject to the state of motion of the frame of reference with respect to which measurements are taken about. Our day-to-day perception of motion is generally based on our earth's view—a view common to all bodies at rest with respect to earth. However, we come across cases when there is a subtle perceptible change in our view of earth. One such case is traveling in the city trains. We easily find that it takes lot longer to overtake another train on a parallel track. Also, we happen to see two people talking while driving separate cars in parallel lanes, as if they were stationary to each other! In terms of kinematics, as a matter of fact, they are actually stationary to each other even though each of them is in motion with respect to ground.

In this topic, we study motion from a perspective other than that of our earth. The only condition that we subject ourselves is that two references or two observers making the measurements of motion of an object, are moving at constant velocity.

We now consider two moving observers, "A" and "B":

The relative velocity of A with respect of B (written as $\vec{\nu}_{AB}$) is $\vec{\nu}_{AB}=\vec{\nu}_A-\vec{\nu}_B$

Similarly, the relative acceleration of A with respect to B is $\vec{a}_{AB}=\vec{a}_A-\vec{a}_B$

**Illustration 9:** Assume that two cars, standing apart, start moving toward each other at speeds of 1 m/s and 2 m/s along a straight road. What could be the speed with which they approach each other?

**(JEE MAIN)**

**Sol:** Let us consider that "A" denotes earth, "B" denotes the first car and "C" denotes the second car. Therefore, the equation of relative velocity for this case is: $\nu_{BA}=1\,m/s$ and $\nu_{CA}=-2\,m/s$.

$$\nu_{CA}=\nu_{BA}+\nu_{CB} \Rightarrow -2=1+\nu_{CB} \Rightarrow \nu_{CB}=-2-1=-3\,m/s$$

This implies that the car "C" is approaching "B" at a speed of –3 m/s along the straight road. Further, it also means that the car "B" is approaching

Relative velocity

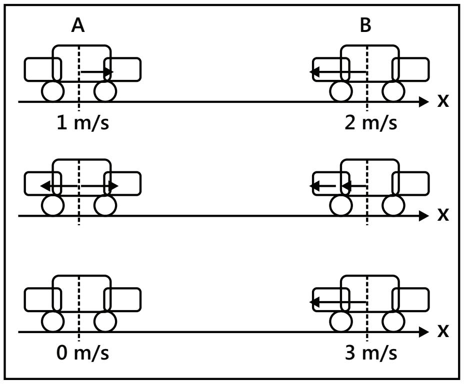



Figure 3.13

"C" at a speed of 3 m/s along the straight road. We, therefore, say that the two cars approach each other at a relative speed of 3 m/s.

To evaluate relative velocity, we proceed as follows:

- Apply velocity of the reference object (say object "A") to other object(s) and hence render the reference object at rest.
- The resultant velocity of the other object ("B") is therefore equal to relative velocity of "B" with respect to "A".

## MASTERJEE CONCEPTS

- The foremost thing in solving problems of relative motion is about visualizing measurement. If we say a body "A" has relative velocity "v" with respect to another moving body "B", then we simply mean that we are making measurement from the moving frame (reference) of "B".
- It is helpful in solving problem to make reference object stationary by applying negative of its velocity to both objects. The resultant velocity of the moving object is equal to the relative velocity of the moving object with respect to reference object. If we interpret relative velocity in this manner, it gives easy visualization as we are accustomed to observing motion from stationary state.

**Nitin Chandrol (JEE 2012, AIR 134)**

**Illustration 10:** Assume that a boy is riding a cycle at a speed of 5√3 m/s toward east along a straight line. It starts raining at a speed of 15 m/s in the vertical direction. What is the direction of rainfall as observed by the boy?

**(JEE MAIN)**

**Sol:** Let us denote earth, boy and rain with symbols A, B and C, respectively. The question here provides the velocity of B and C with respect to A (earth).

$\nu_{BA} = 5\sqrt{3}$ m/s; $\nu_{CA} = 15$ m/s

Now, we need to determine the direction of rain (C) with respect to boy (B), i.e., $\nu_{CB}$. $\nu_{CA} = \nu_{BA} + \nu_{CB} \Rightarrow \nu_{CB} = \nu_{CA} - \nu_{BA}$

Thus, we now draw the vector diagram to evaluate the terms on the right side of the equation. Therefore, here, we need to evaluate "$\nu_{CA} - \nu_{BA}$", which is equivalent to "$\nu_{CA} + (-\nu_{BA})$". We now apply parallelogram theorem to obtain vector sum as represented in the Fig. 3.14 provided.

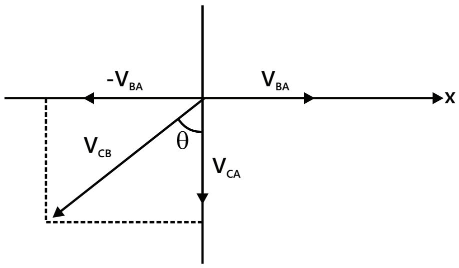



**Figure 3.14**

For the boy (B), the rain appears to fall, making an angle "$\theta$" with the vertical (–y direction).

$$\Rightarrow \tan\theta = \frac{\nu_{BA}}{\nu_{CA}} = \frac{5\sqrt{3}}{15} = \frac{1}{\sqrt{3}} = \tan 30^\circ \Rightarrow \theta = 30^\circ$$

**Illustration 11:** Consider that a person is driving a car toward east at a speed of 80 km/hr. A train appears to move toward north with a velocity of 80√3 km/hr to this person. Find the speed of the train as measured with respect to earth.

**(JEE ADVANCED)**

**Sol:** The velocity of the train with respect to earth is the vector sum of its velocity with respect to car and the velocity of car with respect to earth.

Let us first denote the car and train as "A" and "B," respectively. Here, we are provided with the speed of car ("A") with respect to earth, i.e., "$\nu_A$" and speed of train ("B") with respect to "A," i.e., $\nu_{BA}$. $\nu_A = 80\,\text{km/hr}$; $\nu_{BA} = 80\sqrt{3}\,\text{km/hr}$

Now, we are required to find the speed of train ("B") with respect to earth, i.e., $\nu_B$. From the equation of relative motion, we have

$$\nu_{BA} = \nu_B - \nu_A \Rightarrow \nu_B = \nu_{BA} + \nu_A$$

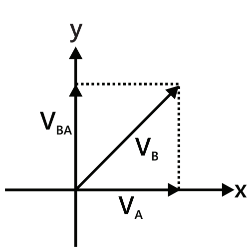



**Figure 3.15**

To evaluate the right-hand side of the equation, we draw vectors " $\nu_{BA}$ " and " $\nu_A$ " and use parallelogram law to find the actual speed of the train.

$$\Rightarrow \nu_B \sqrt{\left\{(\nu_{BA})^2 + (\nu_A)^2\right\}} = \sqrt{\left\{\left(80\sqrt{3}\right)^2 + 80^2\right\}} = 160\,\text{km/hr}$$

## 6.1 Motion of Boat in a Stream

In this section, we consider a general situation of sailing of a boat in a moving stream of water. However, in order to keep our context simplified, we consider that the stream is unidirectional in x-direction and the width of stream, "d", is constant. Let the velocities of boat (A) and stream (B) be" $\nu_A$ " and " $\nu_B$ ," respectively with respect to ground. The velocity of boat (A) with respect to stream (B), therefore, is

$$\nu_{AB} = \nu_A - \nu_B \quad \Rightarrow \nu_A = \nu_{AB} + \nu_B$$

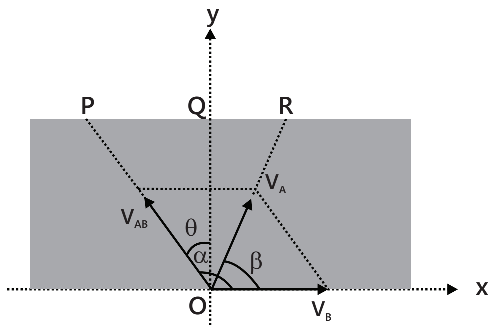


**Figure 3.16**

We represent these velocities in the Fig. 3.16 provided. It is clear from the Fig. 3.16 provided that boat sails in the direction, making an angle " $\theta$ " with y-direction, but reaches destination in different direction. The boat obviously is carried along the stream in x-direction. This displacement in x-direction (x = QR) from the directly opposite position to actual position on the other side of the stream is called the drift of the boat.

### 6.1.1 Resultant Velocity

We can calculate the magnitude of resultant velocity using the parallelogram theorem,

$$\nu_A = \sqrt{\left(\nu_{AB}^2 + \nu_B^2 + 2\nu_{AB}\nu_B\cos\alpha\right)}$$

where " $\alpha$ " is the angle between $\nu_B$ and $\nu_{AB}$ vectors. The angle " $\beta$ " formed by the resultant velocity with x-direction is given as: $\tan\beta = \dfrac{\nu_{AB}\sin\alpha}{\nu_B + \nu_{AB}\cos\alpha}$

### 6.1.2 Time to Cross the Stream

The boat covers a final distance equal to the width of stream "d" in the time "t" in y-direction. Now, by applying the concept of independence of motions in perpendicular directions, we can say that boat covers a final distance "OQ = d" with a speed equal to the component of resultant velocity in y-direction.

Now, the resultant velocity is composed of (i) velocity of boat with respect to stream and (ii) velocity of stream. Here, we observe that velocity of stream is perpendicular to y-direction. Therefore, it does not have any component in y-direction. We, therefore, conclude that the component of the resultant velocity is equal to the component of the velocity of boat with respect to stream in y-direction. Note that the two equal components shown in the Fig. 3.17 provided are geometrically equal as they are altitudes of same parallelogram. Hence, $\nu_{Ay} = \nu_{ABy} = \nu_{AB}\cos\theta$

where " $\theta$ " is the angle that relative velocity of boat w.r.t stream makes with the vertical. $t = \dfrac{d}{\nu_{Ay}} = \dfrac{d}{\nu_{AB}\cos\theta}$

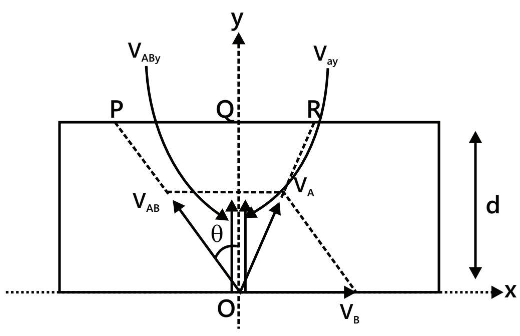


**Figure 3.17**

Thus, we can use either of these two expressions to calculate time to cross the river, depending on the inputs available.

### 6.1.3 Drift of the Boat

We now know that the displacement of the boat in x-direction is independent of motion in the perpendicular direction. Hence, displacement in x-direction is achieved with the component of resultant velocity in x-direction,

$$x = \left(\nu_{Ax}\right)t = \left(\nu_B - \nu_{ABx}\right)t = \left(\nu_B - \nu_{AB}\sin\theta\right)t$$

Then, substituting for time "t", we have: $x = \left(\nu_B - \nu_{AB}\sin\theta\right)\dfrac{d}{\nu_{AB}\cos\theta}$

### 6.1.4 Shortest Interval to Cross the Stream

The time taken by the boat to cross the river is given by: $t = \dfrac{d}{\nu_{Ay}} = \dfrac{d}{\nu_{AB}\cos\theta}$

Clearly, the time taken is minimum for the greatest value of denominator. The denominator is maximum for $\theta = 0°$, for this value, $t_{min} = \dfrac{d}{\nu_{AB}}$

This means that the boat needs to sail in the direction perpendicular to the stream to reach the opposite side in minimum time. The drift of the boat for this condition is: $x = \dfrac{\nu_B d}{\nu_{AB}}$

**MASTERJEE CONCEPTS**

We have discussed motion with specific reference to boat in a water stream. However, the consideration is general and is applicable to the motion of a body in a medium. For example, the discussion and analysis can be extended to the motion of an aircraft, whose velocity is modified by the motion of the wind.

**GV Abhinav (JEE 2012, AIR 329)**

**Illustration 12:** An aircraft flies with velocity of 200 ($\sqrt{2}$)km/hr and the wind is blowing from the south. If the relative velocity of the aircraft with respect to wind is 1000 km/hr, then find the direction in which the aircraft should fly such that it reaches a destination in the north-east direction. **(JEE MAIN)**

**Sol:** The vector sum of the velocity of the airplane with respect to the wind and the velocity of the wind with respect to ground is equal to velocity of the aircraft with respect to ground. This net velocity should be in north-east direction.

We show the velocities pertaining to this problem in the Fig. 3.18 provided. In the Fig. 3.18 provided, OP denotes the velocity of the aircraft in still air or equivalently it represents the relative velocity of the aircraft with respect to air in motion; PQ denotes the velocity of the wind and OQ denotes the resultant velocity of the aircraft. However, it is clear that the aircraft should fly in the direction OP so that it is ultimately led to follow the north-east direction.

We should understand here that one of the velocities is the resultant velocity of the remaining two velocities. Therefore, it follows that the three velocity vectors are represented by the sides of a closed triangle.

We can now demonstrate the direction of OP, if we can find the angle" $\theta$ ". The easiest way to determine the angle between vectors composing a triangle is to apply the sine law,

$$\frac{OP}{\sin 45^\circ} = \frac{PQ}{\sin\theta}$$

Therefore, by substituting these values, we obtain

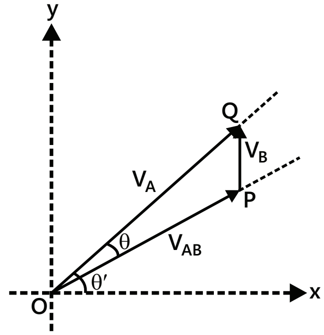



Figure 3.18

$$\sin\theta = \frac{PQ\sin 45^\circ}{OP} = \frac{200\sqrt{2}}{1000x\sqrt{2}} = \frac{1}{5} = 0.2$$

$\theta = \sin^{-1}(0.2)$

Hence, based on the above analysis, the aircraft should steer in the direction, making an angle with east as given by: $\theta' = 45^\circ - \sin^{-1}(0.2)$

**Illustration 13:** Assume that a boat, capable of sailing at 2 m/s, moves upstream in a river. The water in the stream flows at 1 m/s. A person walks from the front to the rear end of the boat at a speed of 1 m/s along the liner direction. What is the speed of the person (m/s) with respect to the ground? **(JEE MAIN)**

**Sol:** First find the velocity of boat with respect to ground. The velocity of man with respect to boat is added to the velocity of boat with respect to ground to get the velocity of man with respect to ground.

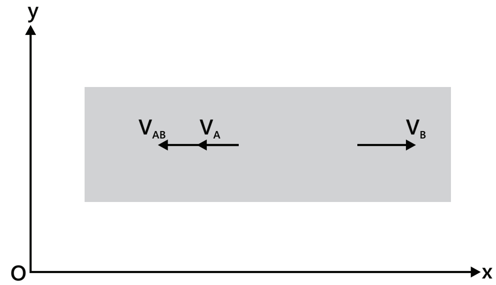


Figure 3.19

Let us assume that the direction of stream be in x-direction and the direction across stream be in y-direction. We further denote boat with "A", stream with "B", and the person with "C". We can now solve this problem in two parts. In the first part, we find out the velocity of boat (A) with respect to ground and then we calculate the velocity of the person (C) with respect to ground.

Here,

velocity of boat (A) with respect to stream (B): $v_{BA}$ = –2 m/s

Velocity of the stream (A) with respect to ground: $v_B$ = 1 m/s

Velocity of the person (C) with respect to boat (A): $v_{CA}$ = 1 m/s

Velocity of the person (C) with respect to ground: $v_C$ = ?

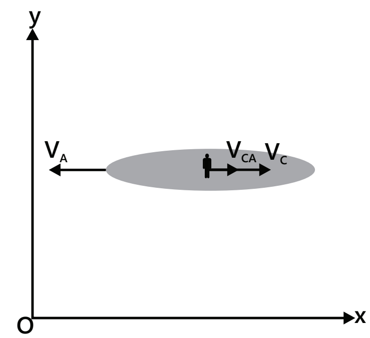


Figure 3.20

The velocity of boat with respect to ground is equal to the resultant velocity of the boat as given by: $v_A = v_{BA} + v_B \Rightarrow v_A = -2 + 1 = -1 m/s$

For the motion of person and boat, the velocity of the person with respect to ground is equal to the resultant velocity of (i) velocity of the person (C) with respect to boat (A) and (ii) velocity of the boat (A) with respect to ground. Hence, $v_C = v_{CA} + v_A \Rightarrow v_C = 1 + (-1) = 0$.

## 7. CIRCULAR MOTION

**Circular motion** is a movement of an object/particle along the circumference of a circle or motion along a circular path. However, it can be uniform or non uniform.

Familiar examples of circular motion include an artificial satellite orbiting the earth at constant height, a stone which is tied to a rope and is being swung in circles and a car turning through a curve in a race track.

**Angular displacement** of a body is the angle in radians (degrees, revolutions) through which a point or line has been turned in a specified sense about a specified axis. Angular displacement is denoted by $\theta$.

The **angular velocity** is defined as the rate of change of angular displacement. The SI unit of angular velocity is radians per second. Angular velocity is usually represented by the symbol omega ($\omega$). $\omega = \frac{d\theta}{dt}, \omega = \frac{v}{r}$ where v is linear velocity.

**Angular acceleration** is the rate of change of angular velocity. In SI units, it is measured in radians per second squared (rad/$s^2$), and is usually denoted by the Greek letter alpha ($\alpha$).

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}, \text{ or } \alpha = \frac{a_T}{r}$$

## 7.1 Uniform Circular Motion

Uniform Circular Motion, involves continuous change in the direction of velocity without any change in its magnitude (v). A change in the direction of velocity is a change in velocity (v). This implies that UCM is associated with acceleration and hence force. Thus, UCM signifies "presence" of force.

In other words, UCM requires a force, which is always perpendicular to the direction of velocity. Since the direction of velocity is continuously changing, the direction of force, being perpendicular to velocity, should also change continuously.

The direction of velocity along the circular trajectory is always tangential in nature. The perpendicular direction to the circular trajectory is, therefore, known as the radial direction. It implies that force (and hence acceleration) in uniform circular motion is radial. For this reason, acceleration in UCM is recognized to require center, i.e., centripetal (seeking center).

Irrespective of whether circular motion is uniform (constant speed) or non-uniform (varying speed), the circular motion inherently associates a radial acceleration to ensure that the direction of motion is continuously changed—at all instants. We learn about the magnitude of radial acceleration soon, but let us be emphatic to differentiate radial acceleration (accounting change in direction that arises from radial force) with tangential acceleration (accounting change in the speed that arises from tangential force).

The coordinates of the particle is given by the x- and y-coordinate pair as: $x = r\cos\theta;\ y = r\sin\theta$

The angle" $\theta$ " is measured anti-clockwise from the x-axis.

The position vector of the position of the particle, r, is represented in terms of unit vectors as:

$$r = x\hat{i} + y\hat{j} \Rightarrow r = r\cos\theta\hat{i} + r\sin\theta\hat{j} \Rightarrow r = r(\cos\theta\hat{i} + \sin\theta\hat{j})$$

The magnitude of velocity of the particle (v) is constant by the definition of UCM. In component form, however, the velocity (refer to the Fig. 3.21) is:

$$v = v_x\hat{i} + v_y\hat{j};\ v_x = -v\sin\theta;\ v_y = v\cos\theta$$

$$\sin\theta = \frac{y}{r};\ \cos\theta = \frac{x}{r};\ v = -\frac{vy}{r}\hat{i} + \frac{vx}{r}\hat{j}$$

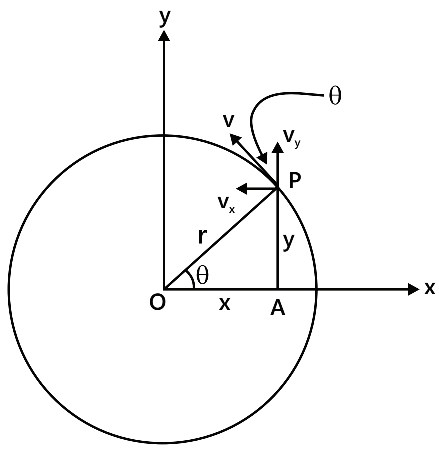


**Figure 3.21**

Acceleration: Knowing that speed, "v" and radius of circle, "r" are constants, we easily differentiate the expression of velocity with respect to time to obtain expression for centripetal acceleration as:

$$a = -\frac{v}{r}\left(\frac{dy}{dt}\hat{i} - \frac{dx}{dt}\hat{j}\right) \Rightarrow a = -\frac{v}{r}\left(v_y\hat{i} - v_x\hat{j}\right)$$

Substituting the value of component velocities in terms of angle, we obtain

$$\Rightarrow a = -\frac{v}{r}\left(v\cos\theta\hat{i} - v\sin\theta\hat{j}\right) = a_x\hat{i} + a_y\hat{j} \qquad \text{where } a_x = -\frac{v^2}{r}\cos\theta;\quad a_y = -\frac{v^2}{r}\sin\theta$$

It is evident from the equation of acceleration that it varies as the angle with horizontal, " $\theta$ " change. Therefore, the magnitude of acceleration is

$$a = |a| = \sqrt{\left(a_x^{\,2} + a_y^{\,2}\right)} \Rightarrow a = |a| = \frac{v}{r}\sqrt{\left\{v^2\left(\cos^2\theta + \cos^2\theta\right)\right\}} \Rightarrow a = \frac{v^2}{r}$$

**Illustration 14:** Assume that a cyclist negotiates the curvature of 20 m at a speed of 20 m/s. What is the magnitude of his acceleration? **(JEE MAIN)**

**Sol:** The speed of the cyclist moving along circular path is constant. So its acceleration is centripetal.

Let the speed of the cyclist be constant. Then, the acceleration of the cyclist is the centripetal acceleration that is required to move the cyclist along a circular path, i.e., the acceleration resulting from the change in the direction of motion along the circular path.

Hence, v = 20 m/s and r = 20 m $\Rightarrow a = \frac{v^2}{r} = \frac{20^2}{20} = 20\,m/s^2$

## 7.2 Non-Uniform Circular Motion

We are aware of the fact that the speed of a particle under circular motion is not constant.

A change in speed means that unequal length of arc (s) is covered in equal time intervals. It further means that the change in the velocity (v) of the particle is not limited to change in direction as in the case of UCM.

Radial or centripetal acceleration. Change in direction is due to radial acceleration (centripetal acceleration), which is given by $a_R = \frac{v^2}{r}$.

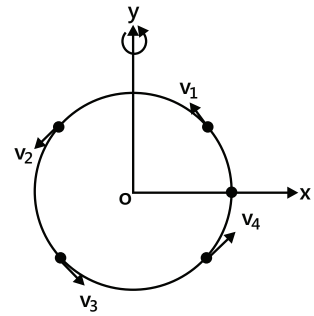



**Figure 3.22**

**Tangential acceleration:** The non-uniform circular motion basically involves a change in speed. This change is accounted by the tangential acceleration, which results due to a tangential force and which acts along the direction of velocity. $a_T = \frac{dv}{dt}$

## 7.3 Relation between Angular and Linear Acceleration

The relationship between angular and linear acceleration is shown hereunder.

$$a_T = \frac{dv}{dt} = \frac{d^2s}{dt^2} = \frac{d^2}{dt^2}(r\theta) = r\frac{d^2\theta}{dt^2} = r\alpha$$

**Illustration 15:** A particle, starting from the position (5 m, 0 m), is moving along a circular path about the origin in x–y plane. The angular position of the particle is a function of time as given here, $\theta = t^2 + 0.2t + 1$. Find (i) tangential acceleration **(JEE MAIN)**

**Sol:** Differentiate the expression for angular position with respect to time to get angular velocity. Tangential acceleration is the product of angular acceleration and the radius.

From the data on initial position of the particle, it is clear that the radius of the circle is 5 m.

(i) For determining tangential acceleration, we need to have expression of linear speed in time.

$$v = \omega r = (2t + 0.2) \times 5 = 10t + 1$$

We obtain tangential acceleration by differentiating the above function: $a_T = \frac{dv}{dt} = 10 m/s^2$

**Illustration 16:** At a particular instant, a particle is moving at a speed of 10 m/s on a circular path of radius 100 m. Its speed is increasing at the rate of 1 m/s$^2$. What is the acceleration of the particle? **(JEE MAIN)**

**Sol:** The acceleration of the particle is the vector sum of the centripetal acceleration and the tangential acceleration. The tangential acceleration is equal to the rate of change of speed.

The acceleration of a particle is the vector sum of mutually perpendicular radial and tangential accelerations. The magnitude of tangential acceleration given here is 1 m/s$^2$. Now, the radial acceleration at the particular instant is:

$$a_R = \frac{v^2}{r} = \frac{10^2}{100} = 1 m/s^2$$

Hence, the magnitude of the acceleration of the particle is: $a = |a| = \sqrt{(a_T^2 + a_R^2)} = \sqrt{1^2 + 1^2} m/s^2 = \sqrt{2} m/s^2$

**Illustration 17:** Which of the following expressions represent the magnitude of centripetal acceleration?:

(A) $\left|\frac{d^2r}{dt^2}\right|$ (B) $\left|\frac{dv}{dt}\right|$ (C) $r\frac{d\theta}{dt}$ (D) None of these **(JEE MAIN)**

**Sol:** The magnitude of centripetal acceleration depends on the square of the magnitude of velocity.

The expression $\left|\frac{d\nu}{dt}\right|$ represents the magnitude of tangential acceleration. The differential $\frac{d\theta}{dt}$ represents the magnitude of angular velocity. The expression $r\frac{d\theta}{dt}$ represents the magnitude of tangential velocity and the expression $\frac{d^2r}{dt^2}$ is second-order differentiation of position vector (r). This is the actual expression of acceleration of a particle under motion. Hence, the expression $\left|\frac{d^2r}{dt^2}\right|$ represents the magnitude of total or resultant acceleration. Hence, option (d) alone is correct.

**Illustration 18:** A particle is executing circular motion. But the magnitude of velocity of the particle changes from zero to (0.3i + 0.4j) m/s in a period of 1 second. The magnitude of average tangential acceleration is:

(A) 0.1 $m/s^2$ (B) 0.2 $m/s^2$ (C) 0.3 $m/s^2$ (D) 0.5 $m/s^2$ **(JEE MAIN)**

**Sol:** Tangential acceleration is equal to the rate of change of speed. Average tangential acceleration is change in speed divided by total time.

The magnitude of average tangential acceleration is the ratio of change in speed and time as given by: $a_T = \frac{\Delta\nu}{\Delta t}$

Now, $\Delta\nu = \sqrt{\left(0.3^2 + 0.4^2\right)} = \sqrt{0.25} = 0.5 m/s;\quad a_T = 0.5 m/s^2$

Hence, option (d) alone is correct.

## MASTERJEE CONCEPTS

Radial acceleration contributes in changing the direction of velocity of an object, but it does not affect the magnitude of velocity. However, tangential acceleration affects the speed of the object in motion.

**Vaibhav Krishan (JEE 2009, AIR 22)**

# FORMULAE SHEET

**(a) Projectile Motion**

Time of flight: $T = \frac{2u\sin\theta}{g}$

Horizontal range: $R = \frac{u^2\sin 2\theta}{g}$

Maximum height: $H = \frac{u^2\sin^2\theta}{2g}$

Trajectory equation (equation of path):

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta} = x\tan\theta\left(1 - \frac{x}{R}\right)$$

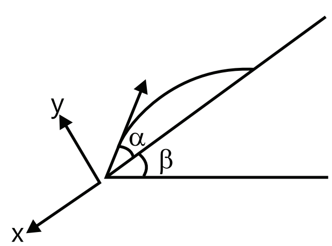


**Figure 3.23**

Projection on an inclined plane

**(b) Relative Motion**

$v_{AB}$ (velocity of A with respect to B) $= v_A - v_B$

$a_{AB}$ (acceleration of A with respect to B) $= a_A - a_B$

Relative motion along straight line $= x_{BA} = x_B - x_A$

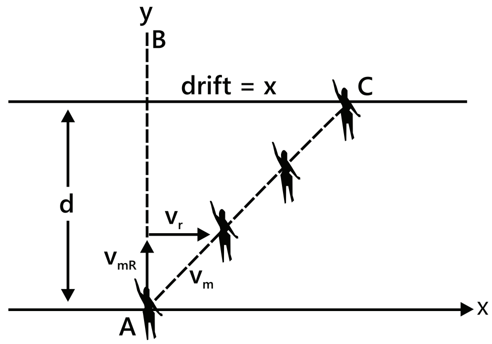


**Figure 3.24**

**(c) Crossing River:** A boat or man in a river always moves in the direction of resultant velocity of velocity of boat (or man) and velocity of the river flow.

**(d) Shortest Time:** Velocity along the river, $V_X = V_R$

Velocity perpendicular to the river, $V_f = V_{mR}$

The net speed is given by $V_m = \sqrt{V_{mR}^2 + V_R^2}$

**(e) Shortest Path:** Velocity along the river, $V_x = 0$

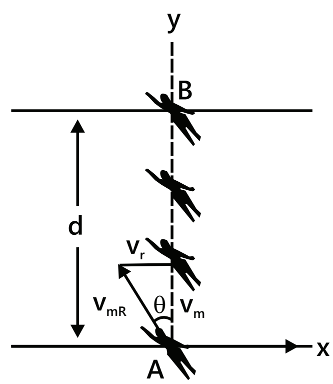


**Figure 3.25**

and velocity perpendicular to river $V_y = \sqrt{V_{mR}^2 - V_R^2}$

The net speed is given by $V_m = \sqrt{V_{mR}^2 - V_R^2}$

at an angle of $90°$ with the river direction.

velocity $V_y$ is used only to cross the river, therefore time to cross the river,

$t = \dfrac{d}{v_y} = \dfrac{d}{\sqrt{v_{mR}^2 - v_R^2}}$ and velocity $v_x$ is zero, therefore, in this case the drift should be zero.

$v_R = v_{mR}\sin\theta = 0$ or $v_R = v_{mR}\sin\theta$ or $\theta = \sin^{-1}\dfrac{v_R}{v_{mR}}$

**(f) Rain Problems:** $v_{Rm} = \vec{v}_R - \vec{v}_m$ or $v_{Rm} = \sqrt{v_R^2 + v_m^2}$

**(g) Circular Motion**

**i.** Average angular velocity $\omega_{av} = \dfrac{\theta_2 - \theta_1}{t_2 - t_1} = \dfrac{\Delta\theta}{\Delta t}$

**ii.** Instantaneous angular velocity $\omega = \dfrac{d\theta}{dt}$

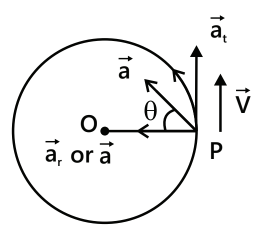


**Figure 3.26**

**iii.** Average angular acceleration $\alpha_{av} = \dfrac{\omega_2 - \omega_1}{t_2 - t_1} = \dfrac{\Delta\omega}{\Delta t}$

**iv.** Instantaneous angular acceleration $\alpha = \dfrac{d\omega}{dt} = \omega\dfrac{d\omega}{d\theta}$

**v.** Relation between speed and angular velocity $v = r\omega$ and $v = \omega\ r$

**vi.** Tangential acceleration (rate of change of speed) $a_t = \dfrac{dV}{dt}$

**vii.** Radial or normal or centripetal acceleration $a_r = \frac{V^2}{r} = \omega^2 r$

**viii.** Total acceleration $\vec{a} = \vec{a}_t + \vec{a}_r, a = \left(a_t^2 + a_r^2\right)^{1/2}$

**ix.** Angular acceleration $\alpha = \frac{d\omega}{dt}$ (non-uniform circular motion)

**x.** Radius of curvature $R = \frac{v^2}{a_\perp} = \frac{mv^2}{F_\perp}$

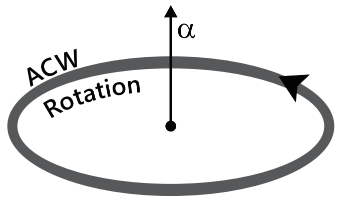



**Figure 3.27**

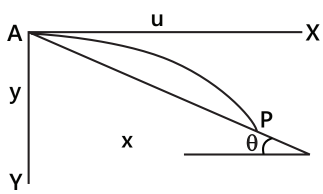
## Solved Examples

### JEE Main/Boards

**Example 1:** A particle is projected horizontally with a speed u from the top of a plane inclined at an angle $\theta$ with the horizontal. How far from the point of projection will the particle strike the plane?

**Sol:** Take the x-axis parallel to the horizontal. Take the y-axis along the vertical. Along x-axis velocity is uniform. Along y-axis initial velocity is zero and acceleration is uniform.

Take, X–Y axes as shown in Figure. Suppose that the particle strikes the plane at point P with coordinates (x and y). Consider the motion between A and P.




Motion in x-direction: initial velocity = u

Acceleration = 0; X = ut ... (i)

Motion in y-direction: initial velocity = 0

Acceleration = g; $y = \frac{1}{2}gt^2$ ... (ii)

Eliminating t from (i) and (ii)

$y = \frac{1}{2}g\frac{x^2}{u^2}$ Also, $y = x\tan\theta$.

Thus, $\frac{gx^2}{2u^2} = x\tan\theta$ giving x = 0, or, $\frac{2u^2\tan\theta}{g}$

Clearly the point P corresponds to x = $\frac{2u^2\tan\theta}{g}$

Then, $y = x\tan\theta = \frac{2u^2\tan^2\theta}{g}$

The distance $AP = l = \sqrt{x^2 + y^2}$

$= \frac{2u^2}{g}\tan\theta\sqrt{1+\tan^2\theta} = \frac{2u^2}{g}\tan\theta\sec\theta$

**Examples 2:** A projectile is projected at an angle 60° from the horizontal with a speed of $\left(\sqrt{3}+1\right)m/s$. The time (in seconds) after which the inclination of the projectile with horizontal becomes 45° is:

**Sol:** Take the x-axis along the horizontal. Take the y-axis vertically upwards. Along x-axis velocity is uniform. Along y-axis initial velocity is positive and acceleration is uniform and negative.

Let "u" and "v" be the speed at the two specified angles. The initial components of velocities in horizontal and vertical directions are:

$u_x = u\cos 60^\circ$

$u_y = u\sin 60^\circ$